We also need to find $E_{S,t}(p_t, N_{S,t})$, the optimal amount of coal, in terms of p_t and $N_{S,t}$. Substituting (20) into (10) for $\bar{e}_S$ and rearranging yields:

$$\bar{e}_S = \alpha p_{S,t}\frac{Y_{S,t}}{E_{S,t}} = \frac{\alpha}{\beta}{p_{S,t}}^{\frac{1}{1-\beta}} N_{S,t} E_{S,t}^{\frac{-(1-\alpha-\beta)}{1-\beta}} \; L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

Then solving this for the coal quantity we have:

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} {p_{S,t}}^{\frac{1}{1-\alpha-\beta}}(p_t) L_{S,t}(p_t) \tag{22}$$

Inserting (22) back into (20) gives:

$$\begin{aligned} Y_{S,t}(p_t, N_{S,t}) &= \frac{1}{\beta} N_{S,t} \; p_{S,t}^{\frac{\beta}{1-\beta}}(p_t) \left[\left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \; {p_{S,t}}^{\frac{1}{1-\alpha-\beta}}(p_t) L_{S,t}(p_t)\right]^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \\ &= \left(\frac{N_{S,t}}{\beta}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{\alpha}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}} {p_{S,t}}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(p_t) L_{S,t}(p_t) \end{aligned} \tag{23}$$

Inserting (19) and (23) into (8) then gives the equilibrium output price ratio in the form:

$$\Gamma^{\sigma} {p_t}^{-\sigma} = \frac{Y_{M,t}(p_t, N_{M,t})}{Y_{S,t}(p_t, N_{S,t})} \tag{36}$$

Lastly, (6), (7), (14), and (15) are the functional forms used in (37)-(38).

Appendix 2: Derivation of Equation (34) for $n_{S,t}$, the Growth Rate of Solow knowledge

The Solow-sector version of (33) is

$$n_{S,t} \equiv \frac{\Delta N_{S,t}}{N_{S,t}} = \; N_{S,t}^{-1} \left(\eta^{\frac{1}{\nu}} \nu (1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(p_{S,t} E_{S,t}^{\alpha} L_{S,t}^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} \tag{A1}$$

and

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}(p_t) \; {p_{S,t}}^{\frac{1}{1-\alpha-\beta}}(p_t) \tag{22}$$

$$\Rightarrow E_{S,t}^{\frac{\alpha}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right)} L_{S,t}^{\frac{\alpha}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} \; {p_{S,t}}^{\frac{\alpha}{1-\beta}\left(\frac{\nu}{1-\nu}\right)\frac{1}{1-\alpha-\beta}}$$

Inserting this into (A1):

$$\begin{aligned} &\Rightarrow n_{S,t} \\ &= N_{S,t}^{-1}\left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} p_{S,t}^{\frac{1}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} \left(\frac{\alpha}{\beta \bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right)} N_{S,t}^{\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right)} L_{S,t}^{\frac{\alpha}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} p_{S,t}^{\frac{\alpha}{1-\beta}\left(\frac{\nu}{1-\nu}\right)\frac{1}{1-\alpha-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{(1-\beta)}\left(\frac{\nu}{1-\nu}\right)} \\ &= \; \left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(\frac{\alpha}{\beta \bar{e}_S}\right)^{\frac{\alpha\nu}{(1-\nu)(1-\alpha-\beta)}} N_{S,t}^{\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right)-1} p_{S,t}^{\frac{\nu}{(1-\nu)(1-\alpha-\beta)}} L_{S,t}^{\frac{\nu}{1-\nu}} \end{aligned} \tag{34}$$

Appendix 3: Signing the slope of the relative supply curve $y^s(N_M, N_S, p)$

To sign this slope, we take the log of $y^s(N_M, N_S, p)$ (39):

$$\ln y^s = constant + \frac{\beta - (\sigma-1)(1-\alpha-\beta)}{1-\beta}\ln p +$$

$$\frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{(\sigma-1)(1-\beta)(1-\alpha-\beta)}\ln\big(1+\Gamma^{\sigma}p^{-(\sigma-1)}\big)$$

where *constant* is the terms that do not depend on *p*. We then take the derivative with respect to *p*, multiply the result by *p*, and simplify:

$$\rho(\sigma) = \frac{\partial \ln y^s}{\partial \ln p} = \frac{\beta - (\sigma-1)(1-\alpha-\beta)}{1-\beta} - \frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{(1-\beta)(1-\alpha-\beta)}\frac{\Gamma^{\sigma}}{p^{\sigma-1}+\Gamma^{\sigma}}$$

We then rearrange the first term:

$$\rho(\sigma) = -\frac{\sigma - 1 - \frac{\beta}{1-\alpha-\beta}}{(1-\beta)/(1-\alpha-\beta)} - \frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{(1-\beta)(1-\alpha-\beta)}\frac{\Gamma^{\sigma}}{p^{\sigma-1}+\Gamma^{\sigma}} \quad \text{(A2)}$$

If $\sigma < 1 + \frac{\beta}{1-\alpha-\beta}\,(< 1 + \frac{1}{1-\alpha-\beta})$ then the first and second terms are both positive and hence $\rho(\sigma) > 0$, so that the relative supply curve is upward sloping.

If $\sigma > 1 + \frac{\beta}{1-\alpha-\beta}$, as $p \to \infty$, $\rho(\sigma) \to -\frac{\sigma-1-\frac{\beta}{1-\alpha-\beta}}{(1-\beta)/(1-\alpha-\beta)} < 0$, and so the curve slopes down. As $p_t \to 0$,

$$\rho(\sigma) \to \frac{\beta-(\sigma-1)(1-\alpha-\beta)}{1-\beta} - \frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{(1-\beta)(1-\alpha-\beta)} = \frac{1}{1-\alpha-\beta} - \sigma \quad \text{(A3)}$$

So, if $\sigma > \frac{1}{1-\alpha-\beta} = 1.905$ for our baseline parameters, the curve slopes down for all p_t; but if $1 + \frac{\beta}{1-\alpha-\beta} < \sigma < \frac{1}{1-\alpha-\beta}$, the curve is backward bending, i.e. sloped upward for low p but downward for high p.

Appendix 4: Proof of Proposition 1 on Comparative Statics

Combining Equations (40) and (41), the derivatives in Proposition 1 in elasticity form are:

$$\partial \ln y_t / \partial \ln \bar{N}_{M,t} = \frac{1}{\rho(\sigma)/\sigma + 1}$$

$$\partial \ln y_t / \partial \ln \bar{E}_M = \frac{\alpha}{1-\beta}\left(\frac{1}{\rho(\sigma)/\sigma + 1}\right)$$

$$\partial \ln y_t / \partial \ln \bar{e}_S = \frac{\alpha}{1-\alpha-\beta}\left(\frac{1}{\rho(\sigma)/\sigma + 1}\right)$$

$$\partial \text{ln}y_t / \partial \text{ln}\bar{N}_{S,t} = -\frac{1-\beta}{1-\alpha-\beta}\left(\frac{1}{\rho(\sigma)/\sigma+1}\right)$$

$$\partial \text{ln}y_t \big/ \partial \text{ln}L_t = -\frac{\alpha}{1-\beta}\left(\frac{1}{\rho(\sigma)/\sigma+1}\right)$$

As we show in the following, $\frac{1}{\frac{\rho(\sigma)}{\sigma}+1} > 0$, hence the signs of the derivatives in Proposition 1 are as shown there. Also, since $1 - \frac{1-\beta}{1-\alpha-\beta} < 0$, the above shows that an equiproportional increase in $\bar{N}_{M,t}$ and $\bar{N}_{S,t}$, i.e. $\Delta \text{ln}(\bar{N}_{M,t}) = \Delta \text{ln}(\bar{N}_{S,t}) > 0$, hence $\Delta \text{ln}(\bar{N}_t) = 0$, results in lower y_t.

To show $\frac{1}{\rho(\sigma)/\sigma+1} > 0$ we need to have $\frac{\rho(\sigma)}{\sigma} > -1$. Dividing $\rho(\sigma)$ from (A2) by σ yields:

$$\frac{\rho(\sigma)}{\sigma} = \frac{\beta-(\sigma-1)(1-\alpha-\beta)}{\sigma(1-\beta)} - \frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{\sigma(1-\beta)(1-\alpha-\beta)}\frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma} \tag{A4}$$

We have $1 > \frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma} > 0$. Evaluating (A4) using the limiting values of $\frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma}$ as p goes to zero or infinity, we have for $\frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma} = 1$:

$$\frac{\big(\beta-(\sigma-1)(1-\alpha-\beta)\big)(1-\alpha-\beta)-\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{\sigma(1-\beta)(1-\alpha-\beta)} = \frac{1}{\sigma(1-\alpha-\beta)} - 1$$

for $\frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma} = 0$:

$$\frac{\beta-(\sigma-1)(1-\alpha-\beta)}{\sigma(1-\beta)} = \frac{1-\alpha}{\sigma(1-\beta)} - \frac{(1-\alpha-\beta)}{(1-\beta)}$$

So, in both cases, and therefore, in intermediate cases as well, $\frac{\rho(\sigma)}{\sigma} > -1$.

Appendix 5: Exploring when an increased elasticity of substitution, σ, decreases $\frac{\rho(\sigma)}{\sigma}$

Here we explore the relevant values of σ, mentioned at the end of Section 4, for which $\frac{\partial(\rho(\sigma)/\sigma)}{\partial\sigma} < 0$. Since $\frac{\partial(\rho/\sigma)}{\partial\sigma} = \frac{\frac{\partial\rho}{\partial\sigma}}{\sigma} - \frac{\rho}{\sigma^2} < 0$ if $\frac{\partial\rho}{\partial\sigma} < 0$, we investigate whether $\frac{\partial\rho}{\partial\sigma} < 0$ for the parameter values and range of p needed for the pre-industrial stagnation analysis in Section 5.2, using (A2) for the supply elasticity $\rho(\sigma) \equiv \frac{\partial \text{ln}y}{\partial \text{ln}p}$, whence:

$$\frac{\partial\rho(\sigma)}{\partial\sigma} = \frac{-(1-\alpha-\beta)}{1-\beta} - \frac{\alpha}{(1-\beta)}\frac{\Gamma^\sigma}{p^{\sigma-1}+\Gamma^\sigma} - \frac{\alpha\big((\sigma-1)(1-\alpha-\beta)-1\big)}{(1-\beta)(1-\alpha-\beta)}\frac{\Gamma^\sigma p^{\sigma-1}(\text{ln}\Gamma-\text{ln}p)}{(p^{\sigma-1}+\Gamma^\sigma)^2} \tag{A5}$$

$\frac{\partial\rho(\sigma)}{\partial\sigma}$ is definitely negative for $p < \Gamma$ and $\sigma > 1 + \frac{1}{1-\alpha-\beta}$ and *vice versa*. These are the relevant conditions for our discussion in Section 5.2 of Pre-industrial Stagnation, hence our conclusion in Section 4 that "for...parameter and price values...relevant to the Pre-industrial Stagnation behavior discussed in Section 5.2...$\frac{\rho(\sigma)}{\sigma}$ decreases...with increasing σ."

We also note that it is negative for a much broader range of values than this as shown by the supply curve shifting from positively sloped to negatively sloped in the previous section. Applying L'Hôpital's rule to (A5) we find:

$$\lim_{p \to \infty} \frac{\partial \rho}{\partial \sigma} = -\frac{1-\alpha-\beta}{1-\beta}, \qquad \lim_{p \to 0} \frac{\partial \rho}{\partial \sigma} = -1$$

As $\rho(\sigma)$ is monotonic in p (see (A2)) if $\rho(\sigma)$ declines sufficiently as σ increases, for example from positive to negative at both extreme values of p, it must also do so for all intermediate values of p. Such changes must also preserve the relevant concavity or convexity properties of (A2). However, this does not preclude $\rho(\sigma)$ locally and temporarily with increasing σ for some intermediate values of p. Evaluating (A5) numerically for different parameter values shows that it is usually negative, though it is possible for σ close to one to get a positive derivative for some range of low values of p. So, there are minor exceptions to our statement above that $\rho(\sigma)/\sigma$ declines with increasing σ.

Appendix 6: Derivation of Equation (45) for $\Delta y_t = 0$ Isoclines in Figures 6a and 6b

Inserting $L_{S,t}(y_t)$ from (14) and $p_{S,t}(y_t)$ from (6) into (22) for coal use:

$$\Longrightarrow E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left[\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\gamma)^{\sigma}\right]^{\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)} \frac{L_t}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}$$

$$\Rightarrow E_t = \frac{\bar{E}_M}{E_{S,t}} = \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \frac{\bar{E}_M}{L_t} \frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}}{(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}} \qquad \text{(B6a)}$$

Substituting this into (21) we have:

$$y_t^{1+\alpha(\sigma-1)} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}} \left(\frac{\bar{E}_M}{L_t}\right)^{\alpha\sigma} \frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\alpha\sigma\left(1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)}}{(1-\gamma)^{\frac{\alpha\sigma^2}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}} \qquad \text{(A6)}$$

Taking logs then differences, and substituting $\Delta y_t / y_t = \Delta\ln(y_t)$ gives (see the Annex at the end):

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}} \Delta\ln(y_t) = \sigma(1-\beta)\ \Delta\ln(N_{M,t})\ -\ \sigma\frac{(1-\beta)^2}{1-\alpha-\beta}\ \Delta\ln(N_{S,t})$$

$$-\ \alpha\sigma\ \Delta\ln(L_t) \qquad \text{(A7)}$$

Using $\Delta\ln(N_{M,t}) = n_t \Delta\ln(N_{S,t}) = n_t \Delta\ln(N_t)/(n_t - 1)$ then gives, after further algebra (again see the Annex):

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}} (n_t - 1)\ \Delta\ln(y_t)$$

$$= \sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) \;-\; (n_t-1)\alpha\sigma\;\Delta\ln(L_t) \tag{A8}$$

With *constant population*, $\Delta\ln(L_t) = 0$, we have from (A8) and (32):

$$\Delta y_t \gtreqless 0 \iff n_t \gtreqless \frac{1-\beta}{1-\alpha-\beta} > 1 \iff N_t \lesseqgtr \left(\frac{1-\alpha-\beta}{1-\beta}\right)^{1-\nu}\Gamma^{\nu} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\nu} \tag{45}$$

with $\Delta y_t = 0$ being below the $\Delta N_t = 0$ isocline as shown in the figures; and since $\frac{1-\beta}{1-\alpha-\beta} > 1$, $\Delta y_t > 0$ below the $\Delta y_t = 0$ isocline and < 0 above it, also as shown. With *population growth*, the $\Delta y_t = 0$ isocline is given by $\sigma(1-\beta)\big(n_t - \frac{1-\beta}{1-\alpha-\beta}\big)\Delta ln(N_t) - (n_t-1)\alpha\sigma\Delta ln(L_t) = 0$, so that:

$$\Delta y_t = 0 \text{ where } n_t > \frac{1-\beta}{1-\alpha-\beta} \text{ and thus } N_t < \left(\frac{1-\alpha-\beta}{1-\beta}\right)^{1-\nu}\Gamma^{\nu} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\nu} \tag{A9}$$

Appendix 7: Proof of Proposition 2 on Existence of Pre-Industrial Stagnation

To prove Proposition 2(a), we first need the following Lemma:

LEMMA. *Given High substitutability, constant population and Assumption 1:*
(i) the locus of all points in (y,N)-space where $\Delta n_t = 0$ is

$$\Gamma^{\frac{1-\alpha-\beta}{\alpha}} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}} N_t^{-\left(\frac{1-\beta}{\alpha}\right)} = \frac{(\sigma-1)(1-\alpha-\beta) - 1 + \left[(\sigma-1)(1-\beta) - \left(\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y^{\frac{\sigma-1}{\sigma}}}{(\sigma-1)(1-2\alpha-\beta) - 1 + \left[(\sigma-1)(1-\alpha-\beta) - \left(\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y^{\frac{\sigma-1}{\sigma}}} \tag{A10}$$

(ii) n falls along the $\Delta n_t = 0$ locus (A10) as it rises in (y, N)-space, and (A10) lies strictly below, and asymptotically (as $y \to \infty$) approaches, the locus defined by

$$\Gamma^{\frac{1-\alpha-\beta}{\alpha}} y^{\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}} N^{-\left(\frac{1-\beta}{\alpha}\right)} = \frac{(1-\beta)(\sigma-\tilde{\sigma})}{(1-\alpha-\beta)(\sigma-\sigma^{\dagger})} \equiv n_{\infty} \tag{A11}$$

Proof of Lemma: (i) We find the $\Delta n_t = 0$ locus in (y, N)-space by taking logs then the differences of (32) with Assumption 1 inserted, $n_t = \Gamma^{\frac{1-\alpha-\beta}{\alpha}} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}} N_t^{-\left(\frac{1-\beta}{\alpha}\right)}$, and then setting $\Delta\ln(n_t) = 0$:

$$0 = \left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}\Delta\ln(y_t) - \frac{1-\beta}{\alpha}\Delta\ln(N_t) \Longrightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)} = \left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{1-\beta} \tag{A12}$$

Substitute this in (A8) with $\Delta\ln(L_t) = 0$, which relates the growth rate of *y* and *N* given constant population, and multiply by $\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)$:

$$\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t-1)\;\Delta\ln(y_t)$$

$$= \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)$$

$$= \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{1-\beta}\Delta\ln(y_t)$$

Divide by $\Delta\ln(y_t)$ and rearrange:

$$\Longrightarrow \left[1+\alpha(\sigma-1)+\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t-1)=\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\alpha-\beta)(\sigma-1)\left(n_t-\tfrac{1-\beta}{1-\alpha-\beta}\right)$$

$$\Longrightarrow \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\alpha-\beta)(\sigma-1)\tfrac{1-\beta}{1-\alpha-\beta}-\left[1+\alpha(\sigma-1)+\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right]=\left[\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\alpha-\beta)(\sigma-1)-\left\{1+\alpha(\sigma-1)+\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right\}\right]n_t$$

The $\Delta n_t = 0$ locus in (y, N)-space is thus:

$$n(y,N)=\Gamma^{\frac{1-\alpha-\beta}{\alpha}}y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}}N_t^{-\left(\frac{1-\beta}{\alpha}\right)}$$

$$=\frac{(\sigma-1)(1-\beta)\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)-\left[1+\alpha(\sigma-1)+\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]}{(\sigma-1)(1-\alpha-\beta)\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)-\left[1+\alpha(\sigma-1)+\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]}$$

$$=\frac{(\sigma-1)(1-\alpha-\beta)-1+\left[(\sigma-1)(1-\beta)-\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y^{\frac{\sigma-1}{\sigma}}}{(\sigma-1)(1-2\alpha-\beta)-1+\left[(\sigma-1)(1-\alpha-\beta)-\left(\tfrac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y^{\frac{\sigma-1}{\sigma}}} \qquad \text{(A10)}$$

(ii) Now by straightforward algebra (see Annex at the end), provided $\sigma > \sigma^{\dagger}$, *n* falls as *y* rises on locus (A10); and from (A10) and (43), the asymptotic lower bound of *n* as $y \to \infty$ is

$$lim_{y\to\infty}n(y,N)=\frac{(\sigma-1)(1-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)}{(\sigma-1)(1-\alpha-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)}=\frac{(1-\beta)(\sigma-\tilde{\sigma})}{(1-\alpha-\beta)(\sigma-\sigma^{\dagger})}\equiv n_{\infty} \qquad \text{(A11)} \ \blacksquare$$

So, the $\Delta n_t = 0$ locus lies below the curve $n(y, N) = n_{\infty}$.

Proof of Proposition 2(a): Since $\tilde{\sigma} < \sigma^{\dagger}$, $n_{\infty} > \frac{(1-\beta)}{(1-\alpha-\beta)}$, hence the curve $n(y,N) = n_{\infty}$ lies below the $\Delta y_t = 0$ locus $n(y, N) = \frac{1-\beta}{1-\alpha-\beta}$; so all development paths below $n(y, N) = n_{\infty}$ have rising y_t. And by Lemma result (ii), *n* falls along the $\Delta n_t = 0$ locus as it rises from left to right in (y, N)-space, hence the locus crosses upwards over curves defined by $n(y, N) =$ constant. By the definition of the locus, all development paths that cross it are locally tangent to the curve with $n(y, N)$ = constant at the point of crossing; and because the locus lies below $n(y, N) = n_{\infty}$, y_t is increasing along those paths. So, all development paths that cross the rising $\Delta n_t = 0$ locus do so from the left, above the locus, to the right, below it, which means they can never rise above the locus later. Such paths therefore have permanently rising y_t, the definition of Pre-industrial Stagnation, and moreover $\Delta n_t > 0$ forever once they cross the $\Delta n_t = 0$ locus. So there is a separatrix in (y, N)-space above the $\Delta n_t = 0$ locus but below the $\Delta y_t = 0$ locus, such that all development paths below the separatrix have forever rising y_t and eventually forever rising n_t, and all development paths above the separatrix eventually cross the $\Delta y_t = 0$ locus, with $\Delta y_t < 0$ thereafter. ■

Proof of Proposition 2(b): We first prove that $\sigma < \sigma^{\dagger}$ (Medium or Low substitutability) means all paths under the $\Delta y_t = 0$ locus in (*y*,*N*)-space eventually rise to cross the locus upwards. For this, we need to show that at any point under this locus, the slope of the path through that point is steeper than the curve $n(y,N) \equiv \Gamma^{\frac{1-\alpha-\beta}{\alpha}}y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{1-\alpha-\beta}{\alpha}}N_t^{-\left(\frac{1-\beta}{\alpha}\right)} = \bar{n}$, where $\bar{n}$ is a constant, through that point. That is, from (A9) and (A8), we need to show that:

$$\left\{\bar{n} > \frac{1-\beta}{1-\alpha-\beta}(>1) \text{ and } \sigma-1 < \frac{1-\beta}{(1-\alpha-\beta)^2}\right\}$$

$$\Longrightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)} = \frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\sigma(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)} + \frac{(\bar{n}-1)\alpha\sigma\ \Delta\ln(L_t)}{\sigma(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(y_t)}$$

$$> \frac{\sigma-1}{\sigma\left(\frac{1-\beta}{1-\alpha-\beta}\right)}$$

Since $\bar{n} > \frac{1-\beta}{1-\alpha-\beta}$, $\Delta\ln(L_t) > 0$ always, and $\Delta\ln(y_t) > 0$ below the $\Delta\ln(y_t) = 0$ locus, the second term on the LHS is > 0. So it will be enough just to prove that

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)} > (\sigma-1)(1-\alpha-\beta) \qquad \text{(A13)}$$

The proof of this by straightforward but tedious algebra is given in the Annex.

Then from (45) and (44), all paths above that isocline eventually cross the $\Delta N_t = 0$ isocline leftwards into the region where $\Delta N_t < 0, \Delta y_t < 0$ forever, as in figure 6a.■

Bounding elasticity value, $\sigma^\dagger$. Finally, we show how from (46) we can derive $\sigma^\dagger$, the threshold value of σ for which the growth of the cost share ratio will first be able to outweigh the diminishing returns to knowledge. Expanding and rearranging (46) and using derivatives from Proposition 1:

$$\Delta\ln n_t = \left(\frac{1-\alpha-\beta}{\alpha}\frac{\sigma-1}{\sigma}\frac{\partial\ln y_t}{\partial\ln N_{M,t}} - \frac{1-\beta}{\alpha}\right)\Delta\ln N_{M,t}$$
$$-\left(\frac{1-\alpha-\beta}{\alpha}\frac{\sigma-1}{\sigma}\frac{1-\beta}{1-\alpha-\beta}\frac{\partial\ln y_t}{\partial\ln N_{M,t}} - \frac{1-\beta}{\alpha}\right)\Delta\ln N_{S,t} \qquad \text{(A14)}$$

Substituting in $\frac{\partial\ln y_t}{\partial\ln N_{M,t}} = \frac{\sigma}{\rho+\sigma}$ from (40) and (41):

$$\Rightarrow \Delta\ln n_t = \left(\frac{1-\alpha-\beta}{\alpha}\frac{\sigma-1}{\rho+\sigma} - \frac{1-\beta}{\alpha}\right)\Delta\ln N_{M,t}$$
$$-\left(\frac{1-\alpha-\beta}{\alpha}\frac{\sigma-1}{\sigma}\frac{1-\beta}{1-\alpha-\beta}\frac{\sigma}{\rho+\sigma} - \frac{1-\beta}{\alpha}\right)\Delta\ln N_{S,t} \qquad \text{(A15)}$$

The necessary conditions for $\Delta\ln n_t > 0$ is that the coefficient of $\Delta\ln N_{M,t}$ here is positive, and if it is close to zero then $\Delta\ln N_{S,t} \to 0$. In the bounding case where we set the coefficient to zero and cross-multiplying and simplifying, we have the condition:

$$(1-\alpha-\beta)(\sigma^\dagger-1) - (1-\beta)(\rho+\sigma^\dagger) = 0$$

Now substituting in the formula for ρ for the case where $p = 0$ from (A3):

$$(1-\alpha-\beta)(\sigma^\dagger-1) - (1-\beta)\frac{1}{1-\alpha-\beta} = 0$$

$$\Rightarrow \sigma^\dagger = 1 + \frac{(1-\beta)}{(1-\alpha-\beta)^2}. \quad ■$$

As σ increases, at first the zone of preindustrial stagnation will appear in the bottom right of figure 6b where very relatively abundant wood results in relative output, *y*, being very high given relative knowledge stocks, *N*.

Appendix 8: Proof of Proposition 3

Proof of Proposition 3(a), Existence of Modern Economic Growth zone

Taking differences of the log of (47) gives $\Delta\ln(y_t) = \sigma[(1-\beta)\Delta\ln(N_t) - \alpha\ \Delta\ln(e_t)]$, and substituting this, and $\Gamma y_t^{\frac{\sigma-1}{\sigma}} = N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}$ from (32), into (A8) gives:

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}\right](n_t-1)[(1-\beta)\Delta\ln(N_t)-\alpha\Delta\ln(e_t)]$$

$$\approx \sigma\left[(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) - (n_t-1)\alpha\Delta\ln(L_t)\right] \quad \text{(A16)}$$

After much further algebra in the Annex at the end, this yields:

$$\left(\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1-\alpha-\beta}\right)(n_t-1)\ \Delta\ln(e_t)$$

$$= \left[\left\{\sigma-\tilde{\sigma}+(\tilde{\sigma}-1)\left(1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}\right)\right\}n_t - (\sigma-\tilde{\sigma})\right]\Delta\ln(N_t) + \left(\frac{1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1-\beta}\right)(n_t-1)\ \Delta\ln(L_t) \quad \text{(A17)}$$

From rearranging (A16) with $\Delta\ln(L_t) = 0$, $\Delta e_t = 0$ when:

$$n_t = \frac{\sigma-\tilde{\sigma}}{\sigma-\tilde{\sigma}+(\tilde{\sigma}-1)\left(1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}\right)}, < 1 \text{ when } \sigma > \tilde{\sigma} \quad \text{(A18)}$$

and given $\sigma > \tilde{\sigma}$, this does have a solution with $0 < n_t < 1$ ($\Delta N_t < 0$) for any permitted parameter values. From (A18), Assumption 1 (35), and (48) (which uses Assumption 1):

$$n_t = \frac{\sigma-\tilde{\sigma}}{\sigma-\tilde{\sigma}+(\tilde{\sigma}-1)(1+N_t^{\frac{1-\beta}{1-\alpha-\beta}}n_t^{\frac{\alpha}{1-\alpha-\beta}})} = \Gamma^{\frac{\sigma(1-\alpha-\beta)}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)} N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}}$$

$$\Rightarrow \frac{\sigma-\tilde{\sigma}}{\Gamma^{\frac{\sigma(1-\alpha-\beta)}{\alpha}}} = N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)}\left[\sigma-1+(\tilde{\sigma}-1)\Gamma^{\sigma} e_t^{-\alpha(\sigma-1)} N_t^{(1-\beta)(\sigma-\tilde{\sigma})+\frac{1-\beta}{1-\alpha-\beta}}\right]$$

Now take total differences, using $(1-\beta)(\sigma-\tilde{\sigma}) + \frac{1-\beta}{1-\alpha-\beta} = (\sigma-1)(1-\beta)$:

$$\Rightarrow 0 =$$

$$\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha} N_t^{\left[\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}\right]-1} e_t^{-(\sigma-1)(1-\alpha-\beta)}\left[\sigma-1 + (\tilde{\sigma}-1)\Gamma^{\sigma} N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)}\right]\Delta N_t$$

$$-(\sigma-1)(1-\alpha-\beta) N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{[-(\sigma-1)(1-\alpha-\beta)]-1}\left[\sigma-1 + (\tilde{\sigma}-1)\Gamma^{\sigma} N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)}\right]\Delta e_t$$

$$+N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)}(\tilde{\sigma}-1)\Gamma^{\sigma}[(\sigma-1)(1-\beta)] N_t^{(\sigma-1)(1-\beta)-1} e_t^{-\alpha(\sigma-1)}\Delta N_t$$

$$-N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)}(\tilde{\sigma}-1)\Gamma^{\sigma}\alpha(\sigma-1) N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)-1}\Delta e_t$$

$$\Rightarrow \left\{ \tfrac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha} N_t^{\left[\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}\right]-1} e_t^{-(\sigma-1)(1-\alpha-\beta)} \left[\sigma - 1 + (\tilde{\sigma} - 1)\Gamma^{\sigma} N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)} (\tilde{\sigma}-1)\Gamma^{\sigma}[(\sigma-1)(1-\beta)] N_t^{(\sigma-1)(1-\beta)-1} e_t^{-\alpha(\sigma-1)} \right\} \Delta N_t$$

$$= \left\{ \tfrac{\alpha(\sigma-1)}{h} N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)-1} \left[\sigma - 1 + (\tilde{\sigma} - 1)\Gamma^{\sigma} N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)} (\tilde{\sigma}-1)\Gamma^{\sigma}\alpha(\sigma-1) N_t^{(\sigma-1)(1-\beta)} e_t^{-\alpha(\sigma-1)-1} \right\} \Delta e_t \qquad \text{(A19)}$$

The bracketed expressions multiplying ΔN_t and Δe_t are both unambiguously positive, so $\Delta N_t / \Delta e_t > 0$, i.e. the isocline is upward sloping. That $\Delta e_t > 0$ above the isocline and < 0 below it then follows from the signs in (A17). ■

Proof of Proposition 3(b) Strong Equilibrium Bias in Modern Economic Growth zone

By part (a), the Modern Economic Growth zone will lie strictly below the $\Delta N_t = 0$ isocline in (e, N)-space, so y_t is falling throughout the zone. Now consider coal use (22) expressed as a function of $(y_t, N_{S,t})$:

$$E_{S,t}(y_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(y_t)\ L_{S,t}(y_t)$$

Substituting $p_{S,t}(y_t)$ from (6) and $L_{S,t}(y_t)$ from (14) converts this to

$$E_{S,t}(y_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} (1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\left(\frac{\sigma-\tilde{\sigma}}{\sigma-1}\right)} L_t, \qquad \text{(A20)}$$

which is rising, since $N_{S,t}$ always rises, y_t is falling, and $\sigma > \tilde{\sigma} > 1$. So we have rising relative coal use $E_{S,t}/\bar{E}_M$ despite a rising relative coal price $\bar{e}_S / e_{M,t}$ (since $e_t \equiv e_{M,t}/\bar{e}_S$ is falling, by definition of a MEG zone).

Appendix 9: Result on Strong Equilibrium Bias in Pre-industrial Stagnation Zone

Because we already know that *e* is increasing unless we are below the $\Delta e = 0$ isocline in figures 7a and 7b, in order to determine where strong bias to wood can happen, we just need to check when coal will decline as p declines. Looking at the demand for coal again:

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)\ L_{S,t}(p_t), \qquad (22)$$

any positive $\Delta N_{S,t}$ increases optimal coal use, *ceteris paribus*. The increasing price of Solow goods, $p_{S,t}$, as the economy becomes more Malthus specialized also will raise coal use, *ceteris paribus*. We therefore need to have a strong enough decline in labor used in the Solow sector to overcome these positive effects and allow coal to decline. For this to happen the elasticity of substitution between the goods must be high enough to allow enough substitution from the Solow good to the Malthus good to allow enough flow of labor between the sectors. Using (6) and (14) we have:

$$p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)\ L_{S,t}(p_t) = (1-\gamma)^{\frac{\sigma}{\sigma-1}} L_t \frac{(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{(\sigma-1)(1-\alpha-\beta)}}}{1+\Gamma^\sigma p_t^{1-\sigma}}$$

$$= \phi(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1-(\sigma-1)(1-\alpha-\beta)}{(\sigma-1)(1-\alpha-\beta)}} \equiv \phi z$$

Next determine $\frac{dz}{dp}$:

$$\frac{dz}{dp} = \frac{1-(\sigma-1)(1-\alpha-\beta)}{(\sigma-1)(1-\alpha-\beta)}(1-\sigma)(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1-2(\sigma-1)(1-\alpha-\beta)}{(\sigma-1)(1-\alpha-\beta)}} \Gamma^\sigma p_t^{-\sigma}$$

If $\sigma > \tilde{\sigma} \equiv 1 + \frac{1}{1-\alpha-\beta}$, then $\frac{dz}{dp} > 0$. So, these terms are declining with declining *p* (and, therefore rising *y*) as long as $\sigma > \tilde{\sigma}$. A minor rearrangement of equation (34) (using Assumption 1, so that $N_{S,t}^{\frac{\alpha}{1-\alpha-\beta}(\frac{\nu}{1-\nu})-1} = 1$):

$$\frac{\Delta N_{S,t}}{N_{S,t}} = \left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(\frac{\alpha}{\beta \bar{e}_S}\right)^{\frac{\alpha\nu}{(1-\nu)(1-\alpha-\beta)}} \left(p_{S,t}^{\frac{1}{1-\alpha-\beta}} L_{S,t}\right)^{\frac{1-\nu}{\nu}} \qquad \text{(A21)}$$

shows that exactly the same term, *z*, drives $\Delta N_{S,t}$. Therefore, if $\sigma > \tilde{\sigma}$ and *p* is declining, $N_{S,t}$ will eventually converge to a constant and coal use can decline.

It is harder to determine over what zone of the phase plane there is strong bias to wood. A simple comparison of the direct and indirect effects of *z* on $E_{S,t}$ in (22) is difficult because the level of *z* affects $E_{S,t}$ both directly and via the change in $N_{S,t}$ through (34). Using the comparative statics framework in Section 4 to find the effect of an exogenous change in $N_{S,t}$ on $E_{S,t}$ will also need to take into account how much $N_{M,t}$ changes, because that also will change the price ratio, *p*.

However, unless $\sigma > \sigma^\dagger$, all paths which initially have falling coal use eventually end up with rising coal use. This is because, initially, the price ratio, *p*, was falling strongly enough to reduce $E_{S,t}$ by more than growing $N_{S,t}$ increases it. But the rate of decline in *p* slows and the level of z remains sufficiently high that through (34) technical change can dominate and increase $E_{S,t}$ again.

We derive equation (51) by first rearranging $e_t(E_t, N_t) = \Gamma^{\sigma/\theta} E_t^{-1/\theta} N_t^{(1-\beta)(\sigma-1)/\theta}$ (50) into $N_t(e_t, E_t) = \Gamma^{\frac{-\sigma}{(1-\beta)(\sigma-1)}} e_t^{\frac{\theta}{(1-\beta)(\sigma-1)}} E_t^{\frac{1}{(1-\beta)(\sigma-1)}}$. Inserting this into $n_t(e_t, N_t) = \Gamma^{\frac{\sigma(1-\alpha-\beta)}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)} N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}}$ (48), and rearranging, gives

$$n_t(e_t, E_t) = \Gamma^{\frac{\sigma}{\alpha(\sigma-1)}} e_t^{\frac{(\sigma-1)(1-2\alpha-\beta)-1}{\alpha(\sigma-1)}} E_t^{\frac{(\sigma-1)(1-\alpha-\beta)-1}{\alpha(\sigma-1)}} \qquad (51)$$

Appendix 10: Proof of Proposition 4 on Asymptotic Growth Rates

Here, we denote growth rates and asymptotic growth rates for variable X_t thus:

$$\frac{\Delta X_t}{X_t} \equiv g(X_t) \text{ and } \lim_{t\to\infty} \frac{\Delta X_t}{X_t} \equiv g_\infty(X_t),$$

with *PS* and *IR* subscripts added as needed.

By definition $y_t \to \infty$ under PS; and by (7) and (15):

$$\lim_{y_t\to\infty} p_{M,t} = \lim_{y_t\to\infty} (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma^{\frac{\sigma}{\sigma-1}} \text{ and}$$

$$\lim_{y_t\to\infty} L_{M,t} = \lim_{y_t\to\infty}\frac{L_t\Gamma}{\Gamma+y_t^{-\frac{\sigma-1}{\sigma}}} = L_\infty$$

and inserting these limits into (52) gives:

$$g_{\infty PS}(N_{M,t}) \equiv \lim_{t\to\infty,PS}\frac{\Delta N_{M,t}}{N_{M,t}} = \lim_{t\to\infty,PS}\lambda_M N_{M,t}^{-1}\left(p_{M,t}\bar{E}_M^{\alpha}L_{M,t}^{1-\alpha-\beta}\right)^{\frac{1-\alpha-\beta}{\alpha(1-\beta)}}$$
$$= \left(\lim_{t\to\infty,PS} N_{M,t}^{-1}\right)\lambda_M^{\frac{1-\alpha-\beta}{\alpha}}\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma^{\frac{\sigma}{\sigma-1}}\right]^{\frac{(1-\alpha-\beta)}{\alpha(1-\beta)}} L_\infty^{\frac{(1-\alpha-\beta)^2}{\alpha(1-\beta)}} = 0 \tag{A22}$$

By definition $y_t \to 0$ under IR, and by (6) and (14),

$$\lim_{y_t\to 0} p_{S,t} = \lim_{y_t\to 0}(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}} \Longrightarrow g_{\infty IR}(p_{S,t}) = 0 \tag{A23}$$

$$\lim_{y_t\to 0} L_{S,t} = \lim_{y_t\to 0}\frac{L_t}{\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1} = L_\infty \tag{A24}$$

and inserting these limits into (53) gives:

$$g_{\infty IR}(N_{S,t}) \equiv \lim_{t\to\infty,IR}\frac{\Delta N_{S,t}}{N_{S,t}} \equiv \lim_{t\to\infty,IR} n_{S,t} \equiv n_{S\infty IR} = \lambda_S(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{\alpha}\right)} L_\infty^{\frac{1-\alpha-\beta}{\alpha}} \tag{A25}$$

Next, we find the growth rates of labor productivity (output per capita) for the Malthus and Solow sectors. For the Malthus sector, substituting (7) for $p_{M,t}(y_t)$ and (15) for $L_{M,t}(y_t)$ into (19) for $Y_{M,t}$ and then rearranging, gives for some constant $\phi_M > 0$ (see the Annex at the end):

$$\frac{Y_{M,t}}{L_{M,t}} = \phi_M N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}} L_t^{\frac{-\alpha}{1-\beta}} \tag{A26}$$

which, using the limits noted above

$$\Longrightarrow g_{\infty PS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = g_{\infty PS}(N_{M,t}) + \frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}0 - \frac{\alpha}{1-\beta}0 = 0 \tag{A27}$$

For the Solow sector, substituting (6) for $p_{S,t}(y_t)$ and (22) for $E_{S,t}$ into (20) for $Y_{S,t}$ and rearranging yields, for some constant $\phi_S > 0$ (again see the Annex):

$$\frac{Y_{S,t}}{L_{S,t}} = \phi_S N_{S,t}^{\frac{1-\beta}{1-\alpha-\beta}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \tag{A28}$$

which by (A25) and other limits

$$\Longrightarrow g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = \frac{1-\beta}{1-\alpha-\beta} n_{S\infty IR} = \left(\frac{1-\beta}{1-\alpha-\beta}\right)\lambda_S(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{\alpha}\right)} L_\infty^{\frac{1-\alpha-\beta}{\alpha}} \tag{55}$$

And since $y_t \to 0$ on an IR path, $L_{S,t} \to L_t$ and (by (1)) $Y_t \to (1-\gamma)^{\frac{\sigma-1}{\sigma}} Y_{S,t}$, hence $g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = g_{\infty IR}\left(\frac{Y_t}{L_t}\right)$, the economy's "growth rate" (i.e. of final output per capita), which by (54) is asymptotically positive. Similar algebra shows that since $y_t \to \infty$ on an PS path, economic growth $g_{\infty PS}\left(\frac{Y_t}{L_t}\right) = g_{\infty PS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = 0$ by (A27). ■

ANNEX

Derivations in Appendix 6 on Derivation of $\Delta y_t = 0$ Isoclines

Steps from (A6) to (A7)

Take logs then differences of (A6):

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\ \Delta\left(\ln(N_{M,t}) - \ln(N_{S,t})\right) - \frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln(N_{S,t})$$

$$-\alpha\sigma\ \Delta\ln(L_t) + \alpha\left(1 - \frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)\frac{\Gamma(\sigma-1)y_t^{\frac{-1}{\sigma}}\Delta y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\ \Delta(\ln N_{M,t} - \ln N_{S,t}) - \frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln(N_{S,t})$$

$$-\alpha\sigma\ \Delta\ln(L_t) + \alpha\left[\sigma - 1 - \left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta y_t/y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

Substituting $\Delta y_t/y_t = \Delta\ln(y_t)$ and rearranging:

$$[1+\alpha(\sigma-1)]\frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\ \Delta\ln(N_{M,t}) - \left[\sigma(1-\beta) + \frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\right]\Delta\ln(N_{S,t})$$

$$-\alpha\sigma\ \Delta\ln(L_t) + \alpha\left[\sigma - 1 - \left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta\ln(y_t)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$\Rightarrow \frac{1+\alpha(\sigma-1) + \left[1+\alpha(\sigma-1) - \alpha(\sigma-1) + \frac{\alpha}{1-\alpha-\beta}\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\ \Delta\ln(N_{M,t}) - \sigma(1-\beta)\ \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_{S,t}) - \alpha\sigma\ \Delta\ln(L_t)$$

$$\Rightarrow \frac{1+\alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\ \Delta\ln(N_{M,t}) - \sigma\ \frac{(1-\beta)^2}{1-\alpha-\beta}\Delta\ln(N_{S,t}) - \alpha\sigma\ \Delta\ln(L_t) \tag{A7}$$

Steps from (A7) to (A8)

To progress from (A7), we need to replace $\Delta\ln(N_{S,t})$, using this:

$$n_t = \frac{\Delta N_{M,t}/N_{M,t}}{\Delta N_{S,t}/N_{S,t}} \Rightarrow n_t\ \Delta\ln(N_{S,t}) = \Delta\ln(N_{M,t}) = \Delta\ln(N_t N_{S,t})$$

$$= \Delta\ln(N_t) + \Delta\ln(N_{S,t})$$

$$\Rightarrow \Delta\ln(N_{S,t}) = \frac{\Delta\ln(N_t)}{n_t - 1} \Rightarrow \Delta\ln(N_{M,t}) = n_t\Delta\ln(N_{S,t}) = \frac{n_t\Delta\ln(N_t)}{n_t - 1}$$

So (A7) becomes:

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$=\sigma(1-\beta)\left(\frac{n_t\Delta\ln(N_t)}{n_t-1}\right)-\sigma\ \frac{(1-\beta)^2}{1-\alpha-\beta}\left(\frac{\Delta\ln(N_t)}{n_t-1}\right)-\alpha\sigma\Delta\ln(L_t)$$

$$=\frac{\sigma(1-\beta)}{n_t-1}\left(n_t-\ \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-\alpha\sigma\Delta\ln(L_t)$$

Multiplying both sides by n_t-1:

$$\Rightarrow\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}(n_t-1)\Delta\ln(y_t)$$
$$=\sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\sigma\Delta\ln(L_t) \quad \text{(A8)}$$

Derivations in Appendix 7 on Existence of Pre-industrial Stagnation

For proof of Lemma part (ii)

To show *n* falls as *y* rises on the $\Delta n_t=0$ locus (A10), write (A10) as

$$n(y,N)=\frac{A+B\Gamma y^{\frac{\sigma-1}{\sigma}}}{C+D\Gamma y^{\frac{\sigma-1}{\sigma}}}\equiv f(\Gamma y^{\frac{\sigma-1}{\sigma}})$$

where $A\equiv(\sigma-1)(1-\alpha-\beta)-1, B\equiv(\sigma-1)(1-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right), C\equiv(\sigma-1)(1-2\alpha-\beta)-1$ and $\mathrm{D}\equiv(\sigma-1)(1-\alpha-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)$. We need to prove that

$$f'\left(y^{\frac{\sigma-1}{\sigma}}\right)=\frac{\left(C+Dy^{\frac{\sigma-1}{\sigma}}\right)B-\left(A+By^{\frac{\sigma-1}{\sigma}}\right)D}{\left(C+Dy^{\frac{\sigma-1}{\sigma}}\right)^2}=\frac{BC-AD}{\left(C+Dy^{\frac{\sigma-1}{\sigma}}\right)^2}<0$$

This is true because we have $\sigma>\sigma^{\dagger}>\tilde{\sigma}=\frac{1}{1-\alpha-\beta}+1$, and hence

$BC-DA$

$$=\left[(\sigma-1)(1-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right][(\sigma-1)(1-2\alpha-\beta)-1]$$
$$-\left[(\sigma-1)(1-\alpha-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right][(\sigma-1)(1-\alpha-\beta)-1]$$
$$=-\alpha\left[(\sigma-1)(1-\beta)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right]+\alpha[(\sigma-1)(1-\alpha-\beta)-1]$$
$$=\alpha\left[-(\sigma-1)(1-\beta)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)+(\sigma-1)(1-\beta)-\alpha(\sigma-1)-1\right]$$
$$=\alpha\left[\frac{\alpha}{1-\alpha-\beta}-\alpha(\sigma-1)\right]=\alpha^2(\tilde{\sigma}-\sigma)<0$$

For proof of Proposition 2(b)

Showing

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)}>(\sigma-1)(1-\alpha-\beta) \quad \text{(A13)}$$

is the same as showing:

$$\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)(1-\alpha-\beta)\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$>\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)(1-\alpha-\beta)-[1+\alpha(\sigma-1)](\bar{n}-1).$$

We prove this inequality is true by showing the [LHS] > 0 and the RHS < 0 as follows:

$$\frac{[\text{LHS}]}{(1-\alpha-\beta)}=\left[\frac{1-\beta}{(1-\alpha-\beta)^2}(\bar{n}-1)-(\sigma-1)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\right]$$
$$=\left(\frac{1-\beta}{(1-\alpha-\beta)^2}-(\sigma-1)\right)\bar{n}-\frac{1-\beta}{(1-\alpha-\beta)^2}+(\sigma-1)\left(\frac{1-\beta}{1-\alpha-\beta}\right)$$

which (because $\sigma-1<\frac{1-\beta}{(1-\alpha-\beta)^2}$ and $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$>\left[\left(\frac{1-\beta}{(1-\alpha-\beta)^2}-(\sigma-1)\right)\frac{1-\beta}{1-\alpha-\beta}+\left(\sigma-1-\frac{1}{1-\alpha-\beta}\right)\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right]$$
$$=\left(\frac{1-\beta}{(1-\alpha-\beta)^2}-\frac{1}{1-\alpha-\beta}\right)\frac{1-\beta}{1-\alpha-\beta}>0.$$

$$\text{RHS}=\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(1-\alpha-\beta)(\sigma-1)-[1+\alpha(\sigma-1)](\bar{n}-1)$$
$$=(1-\alpha-\beta)(\sigma-1)\bar{n}-\bar{n}-\alpha(\sigma-1)\bar{n}-(1-\beta)(\sigma-1)+1+\alpha(\sigma-1)$$
$$=[(1-\alpha-\beta)(\sigma-1)-\alpha(\sigma-1)-1]\bar{n}-(1-\alpha-\beta)(\sigma-1)+1$$

which (again because $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$<[(1-\alpha-\beta)(\sigma-1)-\alpha(\sigma-1)-1]\frac{1-\beta}{1-\alpha-\beta}-(1-\alpha-\beta)(\sigma-1)+1$$
$$=\left[1-\beta-\alpha\frac{1-\beta}{1-\alpha-\beta}-(1-\alpha-\beta)\right](\sigma-1)-\frac{1-\beta}{1-\alpha-\beta}+1$$
$$=-\left(\frac{1-\beta}{1-\alpha-\beta}-1\right)[\alpha(\sigma-1)+1]<0.$$

Derivation in Appendix 8 on Existence of Modern Economic Growth Zone

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}\right](n_t-1)[(1-\beta)\Delta\ln(N_t)-\alpha\Delta\ln(e_t)]$$
$$=\sigma\left[(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\Delta\ln(L_t)\right] \tag{A16}$$

$$\Rightarrow(1-\beta)\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}(n_t-1)-\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Delta\ln(N_t)$$
$$+(n_t-1)\alpha\Delta\ln(L_t)=\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}}n_t^{\frac{1-\nu}{\nu}}}(n_t-1)\alpha\ \Delta\ln(e_t)$$

$$\Rightarrow \left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1-\alpha-\beta}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right)(n_t-1)\alpha\ \Delta\ln(e_t)$$

$$= \left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}} - 1\right) n_t + \frac{1-\beta}{1-\alpha-\beta} - \frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right] \Delta\ln(N_t) + \frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$= \left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}} - 1 - N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right) n_t + \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}} - N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}\right) - 1 - \alpha(\sigma-1)}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right] \Delta\ln(N_t) + \frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$= \left[\left(\frac{\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}} - N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right) n_t + \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right) - 1 - \alpha(\sigma-1)}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right] \Delta\ln(N_t) + \frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

Now substitute $\frac{1-\beta}{1-\alpha-\beta} - 1 = \frac{\alpha}{1-\alpha-\beta} = \alpha(\tilde{\sigma}-1)$, which makes this expression:

$$= \left[\left(\frac{\alpha(\sigma-1)+\alpha(\tilde{\sigma}-1)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right) n_t + \frac{\alpha(\tilde{\sigma}-1)-\alpha(\sigma-1)}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}\right] \Delta\ln(N_t) + \frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1-\alpha-\beta}}{1+N_t^{1+h} n_t^{h}}\right)(n_t-1)\alpha\ \Delta\ln(e_t)$$

$$= \left[\left\{\sigma-1+(\tilde{\sigma}-1)N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}\right\} n_t - (\sigma-\tilde{\sigma})\right] \frac{\alpha\ \Delta\ln(N_t)}{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}} + \frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1-\alpha-\beta}\right)(n_t-1)\ \Delta\ln(e_t)$$

$$= \left[\left\{\sigma-\tilde{\sigma}+(\tilde{\sigma}-1)(1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}})\right\} n_t - (\sigma-\tilde{\sigma})\right]\ \Delta\ln(N_t) + \left(\frac{1+N_t^{\frac{1}{\nu}} n_t^{\frac{1-\nu}{\nu}}}{1-\beta}\right)(n_t-1)\ \Delta\ln(L_t) \quad \text{(A17)}$$

Derivations in Appendix 10 on Asymptotic Growth Rates

Derivation of (A26)

$$(19)\ Y_{M,t} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}} \Longrightarrow \frac{Y_{M,t}}{L_{M,t}} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{-\alpha}{1-\beta}}$$

which, substituting (7) for $p_{M,t}$ and (15) for $L_{M,t}$,

$$= \frac{1}{\beta} N_{M,t} \left[(1-\gamma)^{\frac{\sigma}{\sigma-1}} \Gamma \left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} \left(\frac{L_t \Gamma}{y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma}\right)^{\frac{-\alpha}{1-\beta}}$$

$$= \frac{1}{\beta}[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma]^{\frac{\beta}{1-\beta}}\bar{E}_M^{\frac{\alpha}{1-\beta}}\Gamma^{\frac{-\alpha}{1-\beta}}N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}\left(\frac{\beta}{1-\beta}\right)+\frac{\alpha}{1-\beta}}L_t^{\frac{-\alpha}{1-\beta}}$$

$$= \phi_M N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}}L_t^{\frac{-\alpha}{1-\beta}} \text{ where } \phi_M \equiv \frac{1}{\beta}(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{\beta}{1-\beta}\right)}\bar{E}_M^{\frac{\alpha}{1-\beta}}\Gamma^{\frac{\beta-\alpha}{1-\beta}} > 0 \quad \text{(A26)}$$

Derivation of (A28)

Substituting $E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}\ p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}$ (22) into $Y_{S,t} = \frac{1}{\beta}N_{S,t}p_{S,t}^{\frac{\beta}{1-\beta}}E_{S,t}^{\frac{\alpha}{1-\beta}}L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$ (20) and rearranging:$\Longrightarrow Y_{S,t} = \frac{1}{\beta}N_{S,t}p_{S,t}{}^{\frac{\beta}{1-\beta}}\left[\left(\frac{N_{S,t}}{\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}}L_{S,t}\ p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}\right]^{\frac{\alpha}{1-\beta}}L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$

(Powers: $N_{S,t}$: $\frac{1-\alpha-\beta}{1-\alpha-\beta}+\frac{1-\beta}{1-\alpha-\beta}\frac{\alpha}{1-\beta}=\frac{1-\beta}{1-\alpha-\beta}$; $p_{S,t}$: $\frac{\beta(1-\alpha-\beta)+\alpha}{(1-\beta)(1-\alpha-\beta)}=\frac{\beta(1-\beta)+\alpha(1-\beta)}{(1-\beta)(1-\alpha-\beta)}=\frac{\alpha+\beta}{1-\alpha-\beta}$)

$$= \frac{1}{\beta}\left(\frac{1}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}}p_{S,t}{}^{\frac{\alpha+\beta}{1-\alpha-\beta}}L_{S,t}$$

and then using (6) for $p_{S,t}$

$$\Longrightarrow \frac{Y_{S,t}}{L_{S,t}} = \frac{1}{\beta}\left(\frac{1}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}}\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\left(1+\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

$$= \phi_S N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}}\left(1+\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \quad \text{where } \phi_S \equiv \frac{1}{\beta}\left(\frac{1}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)} > 0 \quad \text{(A28)}$$