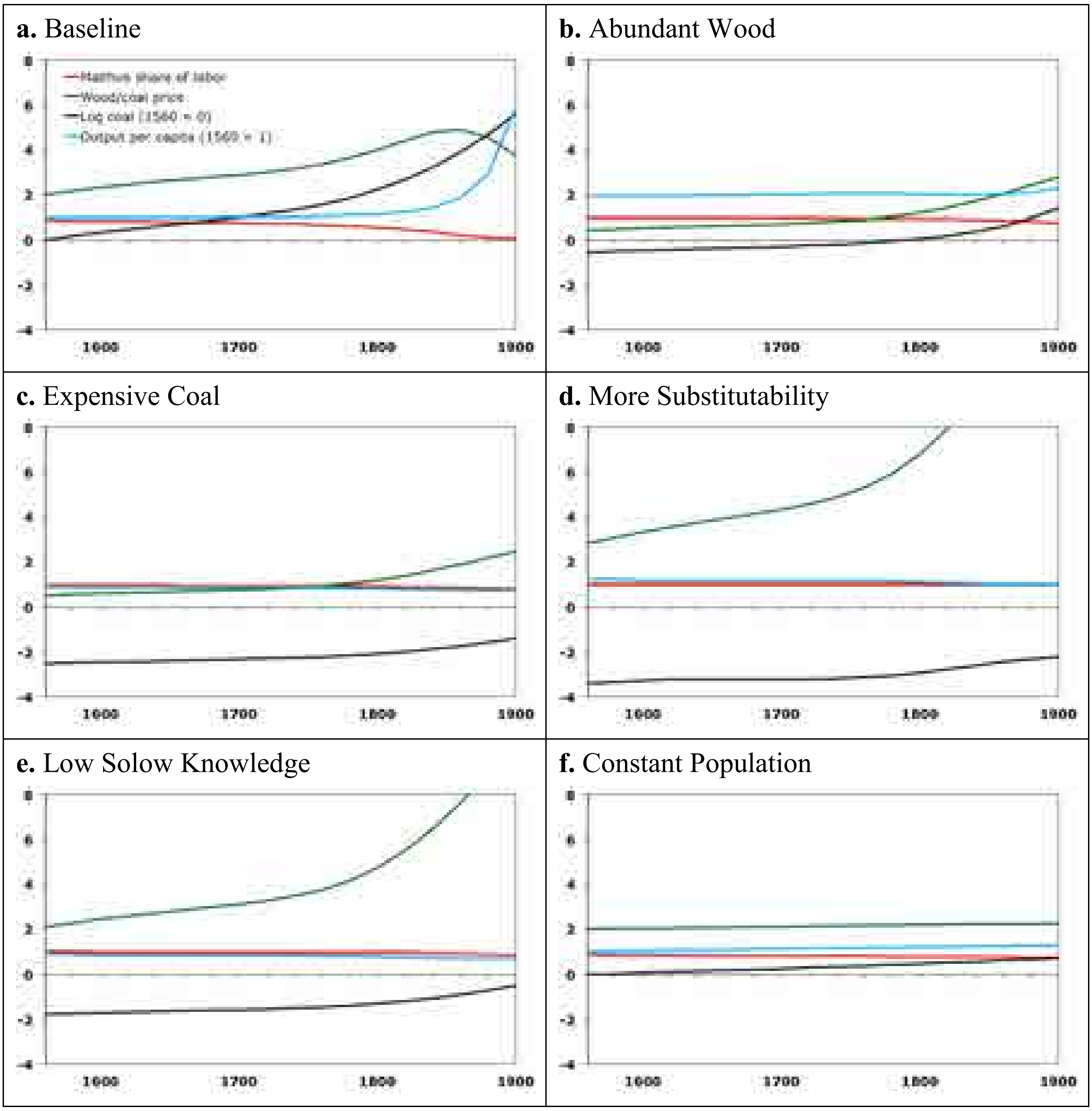
Figure 9. Baseline and Counterfactual Simulations



Notes: The share of labor in the Malthus sector and the wood/coal price are expressed as ratios. The log of coal use is normalized to zero and the level of output per capita to unity in 1560 in the Baseline simulation. Therefore, in Figures 10b-f the levels of coal and output are relative to those in the Baseline simulation.

APPENDIX (FOR ONLINE PUBLICATION)

Appendix A: Derivation of Equilibrium Equations in Section 3

Intermediate goods prices and the labor allocation are jointly determined economy-wide because of the labor adding-up condition (4) and the numeraire equation (5). Then, given goods prices and the labor allocation, all other quantities can be determined for each sector. First, we substitute $p_{M,t} = p_t p_{S,t}$ into the LHS of the numeraire equation (5):

$$\gamma^\sigma \left(p_t p_{S,t}\right)^{1-\sigma} + (1-\gamma)^\sigma p_{S,t}^{1-\sigma} = 1$$

Dividing both sides by $p_{S,t}^{1-\sigma}$ and raising them to the power of $\frac{1}{\sigma-1}$ gives three forms of $p_{S,t}$ for use in (23) and elsewhere (the second and third using $\Gamma \equiv \frac{\gamma}{1-\gamma}$ and $\Gamma^\sigma p_t^{1-\sigma} = \Gamma y_t^{\frac{\sigma-1}{\sigma}}$ from (6)):

$$\begin{aligned} p_{S,t} = [\gamma^\sigma p_t^{1-\sigma} + (1-\gamma)^\sigma]^{\frac{1}{\sigma-1}} &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{\sigma-1}} \\ &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{A1}$$

The price in the Malthus sector for use in (22) is then:

$$\begin{aligned} p_{M,t} = p_t p_{S,t} = (1-\gamma)^{\frac{\sigma}{\sigma-1}} p_t (1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{\sigma-1}} &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}(p_t^{\sigma-1}+\Gamma^\sigma)^{\frac{1}{\sigma-1}} \\ &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{A2}$$

Next we find the optimal levels of labor. Using (9):

$$\begin{aligned} w_t = (1-\alpha-\beta)p_{M,t}\frac{Y_{M,t}}{L_{M,t}} = (1-\alpha-\beta)p_{S,t}\frac{Y_{S,t}}{L_{S,t}} \Longrightarrow \frac{Y_{M,t}}{Y_{S,t}} &= y_t = (\Gamma/p_t)^\sigma \\ = \frac{1}{p_t}\frac{L_{M,t}}{L_{S,t}} \Longrightarrow \frac{L_{M,t}}{L_{S,t}} &= \Gamma^\sigma {p_t}^{1-\sigma} \end{aligned} \tag{A3}$$

Given $L_t = L_{M,t} + L_{S,t}$, $L_{S,t}$ and $L_{M,t}$, for use in (23) and (22) and elsewhere, are given by:

$$L_t = (\Gamma^\sigma {p_t}^{1-\sigma}+1)L_{S,t}(p_t) \Rightarrow L_{S,t} = \frac{L_t}{1+\Gamma^\sigma p_t^{1-\sigma}} = \frac{L_t}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}} \tag{A4}$$

and

$$L_{M,t}(p_t) = L_t - L_{S,t}(p_t) = \frac{L_t \Gamma^\sigma p_t^{1-\sigma}}{\Gamma^\sigma p_t^{1-\sigma}+1} = \frac{L_t\Gamma}{\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}} \tag{A5}$$

Then we substitute the optimal amount of machines sold, $x_{i,t}^*(j)$ from (12), into the goods production functions (2) and (3). Noting that $x_{i,t}^*(j)$ does not vary with j, this yields:

$$Y_{i,t} = \frac{1}{\beta}\left(\int_0^{N_{i,t}} \left(\frac{p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}}{x_{i,t}(j)^{1-\beta}}\right)^{\beta} dj\right)E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta} = \frac{1}{\beta}\left(N_{i,t}(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta})^{\frac{\beta}{1-\beta}}\right)E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}$$

hence

$$Y_{M,t}(p_t, N_{M,t}) = \frac{1}{\beta}N_{M,t}p_{M,t}^{\frac{\beta}{1-\beta}}(p_t)\bar{E}_M^{\frac{\alpha}{1-\beta}}L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{14}$$

and

$$Y_{S,t}(p_t, N_{S,t}) = \frac{1}{\beta}N_{S,t}p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)E_{S,t}^{\frac{\alpha}{1-\beta}}(p_t, N_{S,t})L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{15}$$

We also need to find $E_{S,t}(p_t, N_{S,t})$, the optimal amount of coal, in terms of the endogenous variables p_t and $N_{S,t}$. Substituting (15) into (8) and rearranging yields:

$$\bar{e}_S = \alpha p_{S,t}\frac{Y_{S,t}}{E_{S,t}} = \frac{\alpha}{\beta}p_{S,t}{}^{\frac{1}{1-\beta}}N_{S,t}E_{S,t}^{\frac{-(1-\alpha-\beta)}{1-\beta}}L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

Then solving for the coal quantity we have:

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t) \tag{16}$$

Inserting (16) back into (15) gives:

$$Y_{S,t}(p_t, N_{S,t}) = \frac{1}{\beta}N_{S,t}\, p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)\left[\left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t)\right]^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t)$$

i.e.

$$Y_{S,t}(p_t, N_{S,t}) = \left(\frac{N_{S,t}}{\beta}\right)^{\frac{1-\beta}{1-\alpha-\beta}}\left(\frac{\alpha}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}} p_{S,t}{}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t) \tag{A6}$$

and inserting (A6) and (14) into (6) gives the equilibrium output price ratio in the form:

$$p_t = \Gamma\left(\frac{Y_{M,t}(p_t, N_{M,t})}{Y_{S,t}(p_t, N_{S,t})}\right)^{-\frac{1}{\sigma}} \tag{21}$$

Lastly, (A1)-(A2) and (A4)-(A5) are the functional forms used in (22)-(23).

Appendix B1: Derivation of $n_t(y_t, N_t)$ and $n_t(e_t, N_t)$

Dividing (22) by (23) and using ratio definitions from (24)-(26) gives:

$$n_t \equiv \frac{\Delta N_{M,t}/N_{M,t-1}}{\Delta N_{S,t}/N_{S,t-1}}$$
$$= \left(\frac{p_{M,t}(p_t)}{p_{S,t}(p_t)}\right)^{\frac{1}{m(1-\beta)}} \left(\frac{\bar{E}_{M,t}}{E_{S,t}(p_t, N_{S,t})}\right)^{\frac{\alpha}{m(1-\beta)}} \left(\frac{L_{M,t}(p_t)}{L_{S,t}(p_t)}\right)^{\frac{1-\alpha-\beta}{m(1-\beta)}}$$
$$= p_t^{\frac{1}{m(1-\beta)}} E_t^{\frac{\alpha}{m(1-\beta)}} l_t^{\frac{1-\alpha-\beta}{m(1-\beta)}} \quad \text{(B1)}$$

Equation (B1) is not in the most intuitively useful form, so we substitute $p_t = \Gamma y_t^{-\frac{1}{\sigma}}$ (6) and use other equations, again with ratios defined as in (24)-(26), as follows:

(7)/(8), (6): $e_t = p_t y_t / E_t = \Gamma y_t^{\frac{\sigma-1}{\sigma}} / E_t$ (B2)

(14)/(15), (B1): $y_t = N_t p_t^{\frac{\beta}{1-\beta}} E_t^{\frac{\alpha}{1-\beta}} l_t^{\frac{1-\alpha-\beta}{1-\beta}} = N_t n_t^m / p_t$ (B3)

(9): $y_t = l_t / p_t$ (B4)

Substitute (6) into (B3) and solve for y_t:

$$y_t = N_t n_t^m / \Gamma y_t^{-\frac{1}{\sigma}} \Rightarrow n_t = \Gamma^{\frac{1}{m}} y_t^{\frac{\sigma-1}{m\sigma}} N_t^{\frac{-1}{m}} \quad (28)$$

Substituting in (B3) for p_t from (6), for E_t from (B2), and for l_t from (B4) and (6) gives:

$$y_t = N_t \left(\Gamma y_t^{-\frac{1}{\sigma}}\right)^{\frac{\beta}{1-\beta}} \left(\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{e_t}\right)^{\frac{\alpha}{1-\beta}} \left(\Gamma y_t^{-\frac{1}{\sigma}} y_t\right)^{\frac{1-\alpha-\beta}{1-\beta}} \quad \text{(B5)}$$

Rearranging gives this conversion between (y_t, N_t) and (e_t, N_t) spaces:

$$y_t = \Gamma^{\sigma} e_t^{-\alpha\sigma} N_t^{(1-\beta)\sigma};\ \text{ hence } e_t = \Gamma^{\frac{1}{\alpha}} y_t^{-\frac{1}{\alpha\sigma}} N_t^{\frac{1-\beta}{\alpha}} \quad (29)$$

Inserting (26) for y_t into (25), and using $\tilde{\sigma} \equiv 1 + \frac{1}{1-\beta}$ from (26), finally gives:

$$n_t = \Gamma^{\frac{\sigma}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} N_t^{\frac{(\sigma-1)(1-\beta)-1}{m}} = \Gamma^{\frac{\sigma}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} \quad (30)$$

Appendix B2: Derivation of (33)

Substituting (B2) into (29) gives:

$$y_t = \Gamma^{\sigma} \left(N_t^{(1-\beta)\sigma} \Gamma^{-\alpha\sigma} y_t^{-\alpha\sigma\left(\frac{\sigma-1}{\sigma}\right)} / E_t^{-\alpha\sigma}\right) N_t^{(1-\beta)\sigma} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} y_t^{-\alpha(\sigma-1)} E_t^{\alpha\sigma}$$
$$\Rightarrow y_t^{1+\alpha(\sigma-1)} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} E_t^{\alpha\sigma} \quad \text{(B6)}$$

Inserting (A4) and (A1) into (16):

$$\Rightarrow E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left[\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\gamma)^{\sigma}\right]^{\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)} \frac{L_t}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}}$$

$$\Rightarrow E_t = \frac{\bar{E}_M}{E_{S,t}} = \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \frac{\bar{E}_M}{L_t} \frac{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}}{(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}}$$

Substituting this into (B6) we have:

$$y_t^{1+\alpha(\sigma-1)} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}} \left(\frac{\bar{E}_M}{L_t}\right)^{\alpha\sigma} \frac{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\alpha\sigma\left(1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)}}{(1-\gamma)^{\frac{\alpha\sigma^2}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}} \tag{B7}$$

Taking logs and then differences,[1] and substituting $\Delta y_t / y_t = \Delta \ln(y_t)$ gives (see the Annex):

$$\frac{1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}} \Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right) - \sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right) - \alpha\sigma\,\Delta\ln(L_t) \tag{B8}$$

Using $\Delta\ln\left(N_{M,t}\right) = n_t \Delta\ln\left(N_{S,t}\right) = n_t \Delta\ln(N_t)/(n_t - 1)$ then gives, after further algebra (again see the Annex):

$$\frac{1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}} (n_t - 1)\,\Delta\ln(y_t) = \sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) \; - \; (n_t - 1)\alpha\sigma\,\Delta\ln(L_t) \tag{33}$$

Appendix B3: Derivation of (36)

Taking differences of the log of (29) gives $\Delta\ln(y_t) = \sigma[(1-\beta)\Delta\ln(N_t) - \alpha\,\Delta\ln(e_t)]$, and substituting this, and $\Gamma y_t^{\frac{\sigma-1}{\sigma}} = N_t n_t^m$ from (6) and (B3), into (33) gives:

[1] As noted early in Section 4, first differences in time throughout this paper are treated as if they were differentials, so that all time variables are treated as continuous functions of time.

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right](n_t-1)[(1-\beta)\Delta\ln(N_t)-\alpha\Delta\ln(e_t)]$$
$$\approx\sigma\left[(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\Delta\ln(L_t)\right] \quad \text{(B9)}$$

After much further algebra in the Annex, including using $\frac{1-\beta}{1-\alpha-\beta}-1=\alpha(\sigma^{\dagger}-1)$ from (26), this yields:

$$\left(\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}\right)(n_t-1)\,\Delta\ln(e_t)$$
$$=[\{\sigma-\sigma^{\dagger}+(\sigma^{\dagger}-1)(1+N_t n_t^m)\}n_t-(\sigma-\sigma^{\dagger})]\,\Delta\ln(N_t) \quad (36)$$
$$+\left(\frac{1+Nn_t^m}{1-\beta}\right)(n_t-1)\,\Delta\ln(L_t)$$

Appendix B4: Malthusian Sluggishness under High Substitutability

Proof of Lemma 1

As a preliminary, note that

$$-(1-\beta)<-(1-\alpha-\beta)$$
$$\Rightarrow[(\sigma-1)(1-\alpha-\beta)-1]\,(1-\beta)<[(\sigma-1)(1-\beta)-1]\,(1-\alpha-\beta)$$
$$\Rightarrow\frac{1-\beta}{1-\alpha-\beta}<n_\infty\equiv\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$$

Then rearrange (33) to show that at any path-point, given constant population ($\Delta\ln(L_t)=0$),

$$\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}=\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)}$$

We can then prove (see the Annex for details) that at any point in (y,N)-space on or above the $n=\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}\equiv n_\infty$ locus (38), but not above the $\Delta y_t=0$ locus, i.e. with $\frac{1-\beta}{1-\alpha-\beta}\leq n(y,N)=\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}\leq n_\infty$, the path through that point must have log-slope $\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}>\frac{\sigma-1}{\sigma}$, so that its slope is steeper than the curve $\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}=$ constant through that point. ■

Proof of Lemma 2

(i) We find the $\Delta(n_t)=0$ locus in (y,N)-space by taking logs then the differences of (28) $n_t=\Gamma^{\frac{1}{m}}N_t^{\frac{-1}{m}}y_t^{\frac{\sigma-1}{m\sigma}}$, and then setting $\Delta\ln(n_t)=0$

$$0 = -\frac{1}{m}\Delta\ln(N_t) + \frac{\sigma - 1}{m\sigma}\Delta\ln(y_t) \Rightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)} = \frac{\sigma - 1}{\sigma} \tag{B10}$$

Substitute this in (33) with $\Delta\ln(L_t) = 0$, which relates the growth rate of *y* and *N* given constant population, multiplied by $\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)$:

$$\left(1 + \alpha(\sigma - 1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(n_t - 1)\,\Delta\ln(y_t)$$
$$= \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)$$
$$= \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\frac{\sigma-1}{\sigma}\Delta\ln(y_t)$$

Divide by $\Delta\ln(y_t)$ and rearrange:

$$\Rightarrow \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)(\sigma - 1)$$
$$= \left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t - 1) \tag{B11}$$

Rearranging this, the $\Delta(n_t) = 0$ locus is thus:

$$n(y,N) = \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}}$$
$$= \frac{\frac{1-\beta}{1-\alpha-\beta}\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]}{\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]} \tag{37}$$

$$= \frac{\frac{1-\beta}{1-\alpha-\beta}\left(y^{-\left(\frac{\sigma-1}{\sigma}\right)} + \Gamma\right)(\sigma-1)(1-\beta) - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma + \{1 + \alpha(\sigma-1)\}y^{-\left(\frac{\sigma-1}{\sigma}\right)}\right]}{\left(y^{-\left(\frac{\sigma-1}{\sigma}\right)} + \Gamma\right)(\sigma-1)(1-\beta) - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma + \{1 + \alpha(\sigma-1)\}y^{-\left(\frac{\sigma-1}{\sigma}\right)}\right]} \tag{B12}$$

$$\rightarrow \frac{\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)(1-\beta) - \frac{1-\beta}{1-\alpha-\beta}}{(\sigma-1)(1-\beta) - \frac{1-\beta}{1-\alpha-\beta}} = n_\infty \text{ as } y \rightarrow \infty \tag{38}$$

(ii) We can then show (see the Annex) that for any $0 < y < \infty$, (B12) > (38), i.e. has higher $N^{\frac{-1}{m}}$, i.e. has lower *N*; hence the $\Delta\ln(n_t) = 0$ locus lies below the $n = n_\infty$ locus (38).

(iii) To show that the $\Delta\ln(n_t) = 0$ locus in (N, y)-space is locally steeper than the development path through any point on the locus, we first insert (28) into (B11) and rearrange:

$$\frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} - 1\right)}{\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta)\left(\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} - \frac{1-\beta}{1-\alpha-\beta}\right)} = 1 \tag{B13}$$

The log-slope of this locus in (*y*,*N*) space (again see the Annex) is:

$$\frac{\Delta\ln(N)}{\Delta\ln(y)} = \frac{\left[(1-\alpha-\beta)+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\sigma-\sigma^{\dagger}\right)\frac{n}{m}(\sigma-1)}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$

$$+\frac{\left[\left(\sigma-\sigma^{\dagger}\right)(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$

We can then show (by straightforward but tedious algebra, again in the Annex) that this slope is steeper than the (log-)slope of the path through that point, which since by construction $\Delta\ln(n_t)=0$ at that point, is $\frac{\Delta\ln(N_t)}{\Delta 1(y_t)}=\frac{\sigma-1}{\sigma}$ from (B10). ■

Appendix B5: Proof of upward-sloping $\Delta e_t = 0$ isocline (part-proof of Proposition 2)

$$n_t = \frac{\sigma-\sigma^{\dagger}}{\sigma-\sigma^{\dagger}+(\sigma^{\dagger}-1)(1+N_t n_t^m)} \tag{39}$$

$$n_t = \Gamma^{\frac{\sigma}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} \tag{30}$$

Substitute (30) into (39):

$$\Rightarrow n_t = \frac{\sigma-\sigma^{\dagger}}{\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}} = \Gamma^{\frac{\sigma}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}$$

$$\Rightarrow \frac{\sigma-\sigma^{\dagger}}{\Gamma^{\frac{\sigma}{m}}} = N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]$$

Now take total differences:

$$0 = \frac{(\sigma-\tilde{\sigma})(1-\beta)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}-1} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]\Delta N_t$$

$$-\frac{\alpha(\sigma-1)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}-1}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]\Delta e_t$$

$$+N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[(\sigma-\tilde{\sigma})(1-\beta)+1] N_t^{(\sigma-\tilde{\sigma})(1-\beta)} e_t^{-\alpha(\sigma-1)}\Delta N_t$$

$$-N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[\alpha(\sigma-1)] N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)-1}\Delta e_t$$

$$\Rightarrow \left\{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}-1} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-\right.\right.$$

$$\left.1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[(\sigma-\tilde{\sigma})(1-\beta)+$$

$$\left.1] N_t^{(\sigma-\tilde{\sigma})(1-\beta)} e_t^{-\alpha(\sigma-1)}\right\}\Delta N_t$$

$$= \left\{ \frac{\alpha(\sigma-1)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}-1} \left[\sigma - 1 + (\sigma^{\dagger} - 1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} (\sigma^{\dagger} - 1)\Gamma^{\sigma} \alpha(\sigma - 1) N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)-1} \right\} \Delta e_t$$

The bracketed expressions multiplying_ΔN_t and Δe_t are both unambiguously positive, so $\Delta N_t / \Delta e_t > 0$. ■

Appendix B6 (part-proof of Proposition 3)

To prove that all paths under the $\Delta y_t = 0$ locus in (*y*,*N*)-space eventually rise to cross that locus upwards, we have to show that at any point under this locus, the slope of the path through that point is steeper than the curve $n(y, N) \equiv \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \bar{n}$, where $\bar{n}$ is a constant,, through that point. That is, from (35) and (33), we need to show that:

$$\left\{ \bar{n} > \frac{1-\beta}{1-\alpha-\beta} (> 1) \text{ and } \sigma - 1 < \frac{1}{1-\alpha-\beta} \right\}$$

$$\Rightarrow \frac{\Delta \ln(N_t)}{\Delta \ln(y_t)} = \frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)} + \frac{(\bar{n}-1)\alpha\sigma\, \Delta \ln(L_t)}{\Delta \ln(y_t)} > \frac{\sigma-1}{\sigma}$$

Since $\bar{n} > 1$, $\Delta \ln(L_t) > 0$ always and $\Delta \ln(y_t) > 0$ below the $\Delta \ln(y_t) = 0$ locus, the second term on the LHS is > 0, so it will be enough just to prove that

$$\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)\sigma > \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)$$

$$\text{i.e.} \frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)} > \frac{\sigma-1}{\sigma}$$

The proof of this by straightforward but tedious algebra is given in the Annex. ■

Appendix B7: Growth Rates

Algebra for Proposition 5

We start with these minor rearrangements of (20) for the growth rates of Malthus-sector and Solow-sector machine varieties:

$$n_{M,t} \equiv \frac{\Delta N_{M,t}}{N_{M,t-1}} = \lambda N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{M,t}^{\frac{1}{h(1-\beta)}}(p_t) \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}(p_t) \quad \text{(B15)}$$

$$n_{S,t} \equiv \frac{\Delta N_{S,t}}{N_{S,t-1}} = \lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{S,t}^{\frac{1}{h(1-\beta)}}(p_t) E_{S,t}^{\frac{\alpha}{h(1-\beta)}}(p_t, N_{S,t}) L_{S,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}(p_t) \quad \text{(B16)}$$

where λ is an arbitrary positive constant which may differ from equation to equation. Substituting for coal use $E_{S,t}(p_t, N_{S,t})$ from (16) into (B16) gives after routine algebra (see the Annex) this equation for the growth rate of Solow-sector machine varieties, $n_{S,t}$:

$$n_{S,t} = \frac{\Delta N_{S,t}}{N_{S,t-1}} = \lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{S,t}^{\frac{1}{h(1-\alpha-\beta)}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{h}} \quad \text{(B17)}$$

Substituting for $p_{M,t}^{\frac{1}{m(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{m(1-\beta)}}$ from (A2) and (A5) into (B15) and for $p_{S,t}^{\frac{1}{m(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{m}}$ from (A1) and (A4) into (B17) then yields, after further routine algebra (again see the Annex), these growth rates for each sector:

$$n_{M,t} = \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} \left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{-\left[\frac{(\sigma-\sigma^{\dagger})(1-\alpha-\beta)}{h(\sigma-1)(1-\beta)}\right]} L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} \quad (41)$$

and

$$n_{S,t} = \lambda N_{S,t-1}^{\frac{\nu+\mu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma-\sigma^{\dagger}}{h(\sigma-1)}} L_t^{\frac{1}{m}} \quad (42)$$

Proof of Proposition 5

(i) In (41), all terms are rising or constant: $\Delta N_{M,t-1} > 0$ & $\nu \geq 1-\mu \Rightarrow \Delta N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}} > 0$; $\Delta \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} = 0$; $\Delta y_t > 0$ & $\sigma > \sigma^{\dagger}$ on an MS path $\Rightarrow \Delta\left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{-\left[\frac{(1-\alpha-\beta)(\sigma-\sigma^{\dagger})}{h(\sigma-1)(1-\beta)}\right]} > 0$; and $\Delta L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} > 0$; so $n_{M,t}$, the growth rate of Malthus varieties rises forever.

(ii) In (42), once $\Delta y_t < 0$ forever on an IR path, for $\nu \geq 1-\mu$, all terms are rising or constant for the same reasons as in (i); while for $\frac{1-\alpha-\beta}{1-\beta}(1-\mu) \leq \nu < 1-\mu$, let $\frac{N_{S,t}}{N_{S,t-1}} = k_t (> 1)$ with $\dot{k}_t > 0$; then $\Delta N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} = \Delta k_t^{\frac{1-\mu-\nu}{h\nu}} N_{S,t}^{\frac{(1-\beta)\nu-(1-\mu)(1-\alpha-\beta)}{h\nu(1-\alpha-\beta)}} > 0$: so in either case, $n_{S,t}$, the growth rate of Solow-sector varieties, rises forever. ■

Proof of Proposition 6

In the following, we denote growth rates and asymptotic growth rates for variable X_t thus:

$$\frac{\Delta X_t/\Delta t}{X_t} \equiv g(X_t) \text{ and } \lim_{t\to\infty}\frac{\Delta X_t/\Delta t}{X_t} \equiv g_\infty(X_t),$$

with *MS* and *IR* subscripts added as needed. However, note that $g_{\infty MS}(N_{M,t})$ and $g_{\infty IR}(N_{S,t})$, also written as $n_{M,t\infty MS}$ and $n_{S,t\infty IR}$, are not conventional limits, since we will show that while these growth rates may be signed asymptotically, they generally do not approach fixed limits.

By definition $y_t \to \infty$ under MS; and by (A2) and (A5):

$$\lim_{y_t\to\infty} p_{M,t} = \lim_{y_t\to\infty}(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma^{\frac{\sigma}{\sigma-1}} \Longrightarrow g_{\infty MS}(p_{M,t}) = 0$$

$$\bar{E}_M = \text{constant} \Longrightarrow g_\infty\left(\bar{E}_M^{\frac{\alpha}{h(1-\beta)}}\right) = 0;\ \lim_{y_t\to\infty} L_{M,t} = \lim_{y_t\to\infty}\frac{L_t\Gamma}{\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}} = L_t$$

and inserting these limits into (41) (with $\nu = 1-\mu$ so the $N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}}$ term disappears and h becomes m) gives:

$$g_{\infty MS}(n_{M,t}) = 0 + 0 + g_{\infty MS}\left(L_{M,t}^{\frac{1-\alpha-\beta}{m(1-\beta)}}\right) = \frac{1-\alpha-\beta}{m(1-\beta)}g_\infty(L_t) \tag{B18}$$

By definition $y_t \to 0$ under IR, and by (A1) and (A4),

$$\lim_{y_t\to 0} p_{S,t} = \lim_{y_t\to 0}(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}} \Longrightarrow g_{\infty IR}(p_{S,t}) = 0$$

$$\lim_{y_t\to 0} L_{S,t} = \lim_{y_t\to 0}\frac{L_t}{\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1} = L_t$$

and inserting these limits into (42) gives:

$$g_{\infty IR}(n_{S,t}) = \frac{\alpha}{m(1-\alpha-\beta)}n_{S,t\infty IR} + \frac{1}{m}g_\infty(L_t) \tag{B19}$$

Next we find the growth rates of labor productivity (output per capita) for the Malthus and Solow sectors. For the Malthus sector, substituting (A2) for $p_{M,t}(y_t)$ and (A5) for $L_{M,t}(y_t)$ into (14) for $Y_{M,t}$ and then rearranging, gives (see the Annex):

$$\frac{Y_{M,t}}{L_{M,t}} = \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}}\bar{E}_M^{\frac{\alpha}{1-\beta}}L_t^{\frac{-\alpha}{1-\beta}} \tag{B20}$$

$$\Rightarrow g_{\infty MS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = g_{\infty MS}(N_{M,t}) - \frac{\alpha}{1-\beta} g_{\infty}(L_t)$$
$$= n_{M,t\ \infty MS} - \frac{\alpha}{1-\beta} g_{\infty}(L_t) \quad \text{(B21)}$$

For the Solow sector, substituting (A1) for $p_{S,t}(y_t)$ and (16) for $E_{S,t}$ into (15) for $Y_{S,t}$ and rearranging yields (see the Annex):

$$\frac{Y_{S,t}}{L_{S,t}} = \lambda N_{S,t}^{\frac{1-\beta}{1-\alpha-\beta}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \quad \text{(B22)}$$

$$\Rightarrow g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = \frac{1-\beta}{1-\alpha-\beta} n_{S,t\infty IR} \quad \text{(B23)}$$

Now $y_t \to 0$ on an IR path, $L_{S,t} \to L_t$ and (by (1)) $Y_t \to (1-\gamma)^{\frac{\sigma-1}{\sigma}} Y_{S,t}$, hence $g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = g_{\infty IR}\left(\frac{Y_t}{L_t}\right)$, the economy's "growth rate" (i.e. of final output per capita). So (B23) shows the IR growth rate is eventually a multiple of $n_{S,t\ \infty IR}$, which by (B19) is rising then, i.e. growth eventually accelerates on an IR path. Similar algebra shows that since $y_t \to \infty$ on an MS path, $g_{\infty MS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = g_{\infty MS}\left(\frac{Y_t}{L_t}\right)$. So (B21) shows that under MS, economic growth is eventually slower than $n_{M,t}$, the growth rate of machine varieties on that path, because of the drag of population growth at rate $g_{\infty}(L_t)$. From the asymptotic growth rates in (B18) and (B19), for any pair of MS and IR paths starting with the same parameters except for different initial varieties $(N_{M,0}, N_{S,0})$ (a difference needed to make one path MS and the other IR), after some finite time eventually $n_{S,t}$ on the IR path must exceed $n_{M,t}$ on the MS path. Hence $g_{\infty IR}\left(\frac{Y_t}{L_t}\right) > g_{\infty MS}\left(\frac{Y_t}{L_t}\right)$: economic growth is eventually at on an IR path than on an MS path.

■

Appendix B8: Proof of Propositions 7 and 8

If we fix $N_{M,t}$ and $N_{S,t}$ – i.e. treat them as constants – the equation system consists of $p_t = \Gamma\left(\frac{Y_{M,t}(p_t,N_{M,t})}{Y_{S,t}(p_t,N_{S,t})}\right)^{-\frac{1}{\sigma}}$ (21) alone, or even simpler from (6), $y_t = \frac{Y_{M,t}(y_t,N_{M,t})}{Y_{S,t}(y_t,N_{S,t})}$.

Comparative static effects on y_t

Substitute (14) for $Y_{M,t}(p_t, N_{M,t})$ and (A6) for $Y_{S,t}(p_t, N_{S,t})$ into the latter equation:

$$y_t = \frac{Y_{M,t}(y_t, N_{M,t})}{Y_{S,t}(y_t, N_{S,t})} = \frac{N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}}(y_t) \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(y_t)}{N_{S,t}^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(y_t) \left(\frac{\alpha}{\beta \bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}} L_{S,t}(y_t)} \tag{B24}$$

Take logs, let $\mathbf{\Omega}$ be the vector of all model parameters, and momentarily fix $N_{M,t}$ and $N_{S,t}$, which we then denote as $\bar{N}_{M,t}$ and $\bar{N}_{S,t}$. This gives:

$$\begin{aligned} f(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}) \equiv \ln(\bar{N}_{M,t}) - \left(\frac{1-\beta}{1-\alpha-\beta}\right) \ln(\bar{N}_{S,t}) \\ + \frac{\beta}{1-\beta} \ln\left(p_{M,t}(y_t)\right) - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right) \ln\left(p_{S,t}(y_t)\right) \\ + \left(\frac{\alpha}{1-\beta}\right) \ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right) \ln\left(\frac{\alpha}{\beta \bar{e}_S}\right) \\ + \left(\frac{1-\alpha-\beta}{1-\beta}\right) \ln\left(L_{M,t}(y_t)\right) - \ln\left(L_{S,t}(y_t)\right) - \ln(y_t) = 0 \end{aligned} \tag{B25}$$

We next calculate $\partial f / \partial y_t$:

$$\begin{aligned} \frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right) \frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right) \frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} \\ + \left(\frac{1-\alpha-\beta}{1-\beta}\right) \frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t} - \frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} - \frac{1}{y_t} \end{aligned} \tag{B26}$$

We find the four partial derivatives in (B26) as follows. From (A1), taking logs and then the derivative (see the Annex for the full algebra):

$$\frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} = \frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}} \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} > 0$$

From (A2) and the previous result:

$$\frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} = \frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0$$

From (A4):

$$\frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} = \frac{-(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0$$

From (A5) and the previous result: $\frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t} = \frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} > 0$

Hence (B26) becomes:

$$\frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right)\frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} + \frac{(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)} - \frac{1}{y_t}$$

which after several lines of algebra (see the Annex) simplifies to:

$$\frac{\partial f}{\partial y_t} = -\frac{(1-\alpha-\beta)(1-\alpha+\sigma\alpha)+(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\sigma(1-\beta)(1-\alpha-\beta)y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0 \qquad \text{(B27)}$$

Lastly, we calculate $\partial f\left(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right)/\partial \Omega_j$ from (B25) for selected parameters Ω_j, and insert the results and $\partial f/\partial y_t < 0$ from (B27) into $\frac{\partial y_t}{\partial \mathbf{\Omega}} \approx -\frac{\partial f(y_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega})/\partial \mathbf{\Omega}}{\partial f/\partial y_t}$ and $\frac{\partial y_t}{\partial \bar{N}_{i,t}} = -\frac{\partial f(y_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega})/\partial \bar{N}_{i,t}}{\partial f/\partial y_t}$ from the implicit function theorem, to yield the results shown in Proposition 7 :

$$\partial y_t/\partial \bar{N}_{M,t} = -\frac{1}{\bar{N}_{M,t}(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial \bar{N}_{S,t} = \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)}{\bar{N}_{S,t}(\partial f/\partial y_t)} < 0$$

$$\partial y_t/\partial \bar{E}_M = -\frac{\left(\frac{\alpha}{1-\beta}\right)}{\bar{E}_M(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial \bar{e}_S = -\frac{\left(\frac{\alpha}{1-\alpha-\beta}\right)}{\bar{e}_S(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial L_t = -\frac{\left(\frac{1-\alpha-\beta}{1-\beta}\right)-1}{L_t(\partial f/\partial y_t)} < 0$$

And since (B25) $\Rightarrow \frac{\partial y_t}{\partial \ln(\bar{N}_{S,t})} = \left(\frac{1-\beta}{1-\alpha-\beta}\right)\frac{\partial y_t}{\partial \ln(\bar{N}_{M,t})} < 0$, we also have that equal increases in $\ln(\bar{N}_{M,t})$ and $\ln(\bar{N}_{S,t})$, i.e. a larger total number of varieties $\bar{N}_{M,t} + \bar{N}_{S,t}$ which leaves the ratio $\bar{N}_t$ unchanged, will increase industrialisation (lower y_t). ■

Comparative static effects on the energy price ratio, e_t

Finding the effects of parameters on e_t starts by transforming (29) with (25) into $\ln(y_t) = \sigma\ln(\Gamma) + (1-\beta)\sigma\left(\ln(N_{M,t}) - \ln(N_{S,t})\right) - \alpha\sigma\ln(e_t)$ and using this to substitute for $\ln(y_t)$ in equation (B25) with dependencies of e_t. After much algebra (see the Annex) this yields:

$$\begin{aligned} & f(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}) \\ &= -\sigma\ln(\Gamma) + [1-(1-\beta)\sigma]\ln(\bar{N}_{M,t}) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln(\bar{N}_{S,t}) \\ &\quad - \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] \\ &\quad + \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\ &\quad - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \qquad \text{(B28)} \\ &\quad + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\ &\quad + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma) + (\sigma-1)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\} \\ &\quad - \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) - \ln(L_t) \\ &\quad + \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \alpha\sigma\ln(e_t) \end{aligned}$$

Equation (29), in the form $\ln(y_t) = \sigma\ln(\Gamma) + (1-\beta)\sigma\ln(\bar{N}_t) - \alpha\sigma\ln(e_t)$, means that $\frac{\partial f}{\partial e_t} = -\alpha\sigma\frac{\partial f}{\partial y_t} > 0$. This, combined with the only two tractable partial derivatives from (B28), gives the comparative static results shown in Proposition 8:

$$\partial e_t/\partial \bar{E}_M = -\frac{\left(\frac{\alpha}{1-\beta}\right)}{\bar{E}_M(\partial f/\partial e_t)} < 0$$

$$\partial e_t/\partial L_t = -\frac{\left(\frac{1-\alpha-\beta}{1-\beta}\right)-1}{L_1(\partial f/\partial e_t)} > 0$$

Lastly, since (B28) $\Rightarrow \left.\frac{\partial e_t}{\partial \ln(\bar{N}_{M,t})}\right|_{\bar{N}_t \text{ constant}} = \frac{\frac{1-\beta}{1-\alpha-\beta}-(1-\beta)\sigma}{1-(1-\beta)\sigma} \left.\frac{\partial e_t}{\partial \ln(\bar{N}_{S,t})}\right|_{\bar{N}_t \text{ constant}} > 0$, we also have that equal rises in $\ln(\bar{N}_{M,t})$ and $\ln(\bar{N}_{S,t})$, i.e. a larger total number of varieties $\bar{N}_{M,t} + \bar{N}_{S,t}$ which keeps the ratio $\bar{N}_t$ constant, will raise the energy price ratio e_t. ■

ANNEX (FOR ONLINE PUBLICATION)

Derivations in Appendix B2

Steps from (B7) to (B8)

Take logs then differences of (B7):

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\left(\ln\left(N_{M,t}\right)-\ln\left(N_{S,t}\right)\right) - \frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t) + \alpha\left(1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)\frac{\Gamma(\sigma-1)y_t^{\frac{-1}{\sigma}}\Delta y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\left(\ln N_{M,t}-\ln N_{S,t}\right) - \frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t) + \alpha\left[\sigma-1-\left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta y_t/y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

Substituting $\Delta y_t/y_t = \Delta\ln(y_t)$ and rearranging:

$$[1+\alpha(\sigma-1)]\frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right) - \left[\sigma(1-\beta)+\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\right]\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t) + \alpha\left[\sigma-1-\left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta\ln(y_t)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left[1+\alpha(\sigma-1)-\alpha(\sigma-1)+\frac{\alpha}{1-\alpha-\beta}\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right) - \sigma(1-\beta)\,\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln\left(N_{S,t}\right) - \alpha\sigma\,\Delta\ln(L_t)$$

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right)-\sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right)-\alpha\sigma\,\Delta\ln(L_t) \quad \text{(B8)}$$

Steps from (B8) to (27)

To progress from (B8), we need to replace $\Delta\ln\left(N_{S,t}\right)$, using this:

$$n_t = \frac{\Delta N_{M,t}/N_{M,t}}{\Delta N_{S,t}/N_{S,t}} \Rightarrow n_t\,\Delta\ln\left(N_{S,t}\right)=\Delta\ln\left(N_{M,t}\right)=\Delta\ln\left(N_t N_{S,t}\right)=\Delta\ln(N_t)+\Delta\ln\left(N_{S,t}\right)$$

$$\Rightarrow \Delta\ln\left(N_{S,t}\right)=\frac{\Delta\ln(N_t)}{n_t-1}\Rightarrow \Delta\ln\left(N_{M,t}\right)=n_t\Delta\ln\left(N_{S,t}\right)=\frac{n_t\Delta\ln(N_t)}{n_t-1}$$

So (B8) becomes:

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t) = \sigma(1-\beta)\left(\frac{n_t\Delta\ln(N_t)}{n_t-1}\right)-\sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\left(\frac{\Delta\ln(N_t)}{n_t-1}\right)-\alpha\sigma\Delta\ln(L_t)$$

$$=\frac{\sigma(1-\beta)}{n_t-1}\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-\alpha\sigma\Delta\ln(L_t)$$

Multiplying both sides by n_t-1:

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}(n_t-1)\Delta\ln(y_t) = \sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\sigma\Delta\ln(L_t) \quad \text{(27)}$$

Derivations in Appendix B3

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right](n_t-1)[(1-\beta)\Delta\ln(N_t) - \alpha\Delta\ln(e_t)] = \sigma\left[(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\Delta\ln(L_t)\right] \quad \text{(B9)}$$

$$\Rightarrow (1-\beta)\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}(n_t-1)-\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Delta\ln(N_t)$$
$$+(n_t-1)\alpha\Delta\ln(L_t)$$
$$=\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}(n_t-1)\alpha\,\Delta\ln(e_t)$$

$$\Rightarrow\left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}}{1+N_t n_t^m}\right)(n_t-1)\alpha\,\Delta\ln(e_t)$$
$$=\left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}-1\right)n_t+\frac{1-\beta}{1-\alpha-\beta}\right.$$
$$\left.-\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right]\Delta\ln(N_t)+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$=\left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m-1-N_t n_t^m}{1+N_t n_t^m}\right)n_t\right.$$
$$\left.+\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)(1+N_t n_t^m-N_t n_t^m)-1-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$
$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$=\left[\left(\frac{\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m-N_t n_t^m}{1+N_t n_t^m}\right)n_t+\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)-1-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$
$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

Now substitute $\frac{1-\beta}{1-\alpha-\beta}-1=\frac{\alpha}{1-\alpha-\beta}=\alpha(\sigma^{\dagger}-1)$, which makes this expression:

$$=\left[\left(\frac{\alpha(\sigma-1)+\alpha(\sigma^{\dagger}-1)N_t n_t^m}{1+N_t n_t^m}\right)n_t+\frac{\alpha(\sigma^{\dagger}-1)-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$
$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}}{1+N_t n_t^m}\right)(n_t-1)\alpha\,\Delta\ln(e_t)$$

$$= [\{\sigma-1+(\sigma^\dagger-1)N_t n_t^m\}n_t-(\sigma-\sigma^\dagger)]\frac{\alpha\,\Delta\ln(N_t)}{1+N_t n_t^m}+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}\right)(n_t-1)\,\Delta\ln(e_t)$$

$$= [\{\sigma-\sigma^\dagger+(\sigma^\dagger-1)(1+N_t n_t^m)\}n_t-(\sigma-\sigma^\dagger)]\,\Delta\ln(N_t) \qquad (29)$$

$$+\left(\frac{1+N_t n_t^m}{1-\beta}\right)(n_t-1)\,\Delta\ln(L_t)$$

Derivations in Appendix B4

Algebra for Lemma 1

We need to show that:

$$\frac{1-\beta}{1-\alpha-\beta}\le \bar{n}\le \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1} \text{ and } \sigma>\sigma^\dagger, \text{i.e.}\,\sigma-1>\frac{1}{1-\alpha-\beta},$$

$$\Rightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}=\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)}>\frac{\sigma-1}{\sigma}.$$

$$\text{Well, this inequality } \Leftrightarrow \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)$$

$$>\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow [1+\alpha(\sigma-1)](\bar{n}-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}(\bar{n}-1)$$

$$>(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$+\Gamma y_t^{\frac{\sigma-1}{\sigma}}(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)\Gamma y_t^{\frac{\sigma-1}{\sigma}}-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$

$$>(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

$$\Leftrightarrow \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$> (\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

So we can prove the $\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}$ inequality true by showing [LHS] > 0 and RHS < 0 as follows:

$$[\text{LHS}] = \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

which (because $\sigma-1>\frac{1}{1-\alpha-\beta}$)

$$> \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$= \left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\frac{1-\beta}{1-\alpha-\beta}-1\right) > 0$$

$$\text{RHS} = (\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

$$= [(\sigma-1)(1-\alpha-\beta)-1]\bar{n}-(\sigma-1)(1-\beta)\frac{1-\beta}{1-\alpha-\beta}+1+\alpha(\sigma-1)$$

which (because $\bar{n} \le \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$)

$$\le [(\sigma-1)(1-\alpha-\beta)-1]\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}+(\sigma-1)\frac{\alpha(1-\alpha-\beta)-(1-\beta)^2}{1-\alpha-\beta}+1$$

$$= (\sigma-1)(1-\beta)-1-(\sigma-1)\frac{\alpha^2+(1-\alpha-\beta)(1-\beta)}{1-\alpha-\beta}+1$$

$$= (\sigma-1)(1-\beta)-1-\frac{(\sigma-1)\alpha^2}{1-\alpha-\beta}-[(\sigma-1)(1-\beta)-1] = -\frac{(\sigma-1)\alpha^2}{1-\alpha-\beta} < 0.$$

Detailed proofs for Lemma 2

(ii) Proof that (B12) > (38), i.e. that

$$\frac{\frac{1-\beta}{1-\alpha-\beta}\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)-\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right]}{\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)-\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right]}$$
$$> \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$$

Now let $\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta) \equiv W,\ \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}} \equiv X$

then we must show that: $\dfrac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)W-X}{W-X} > \dfrac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$

$$\Leftrightarrow \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)W-X\right][(\sigma-1)(1-\alpha-\beta)-1] > (W-X)[(\sigma-1)(1-\beta)-1]$$

$$\Leftrightarrow (\sigma-1)(1-\beta)W-(\sigma-1)(1-\alpha-\beta)X-\left(\frac{1-\beta}{1-\alpha-\beta}\right)W+X$$
$$> (\sigma-1)(1-\beta)W-W-(\sigma-1)(1-\beta)X+X$$

$$\Leftrightarrow -(\sigma-1)(1-\alpha-\beta)X-\left(\frac{1-\beta}{1-\alpha-\beta}\right)W > -W-(\sigma-1)(1-\beta)X$$

$$\Leftrightarrow (\sigma-1)\alpha X > \left(\frac{1-\beta}{1-\alpha-\beta}-1\right)W = \left(\frac{\alpha}{1-\alpha-\beta}\right)W$$

$$\Leftrightarrow (\sigma-1)X > \frac{W}{1-\alpha-\beta}$$

$$\Leftrightarrow (\sigma-1)\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right] > \frac{\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)}{1-\alpha-\beta}$$

$$\Leftrightarrow \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}} > y^{-\frac{\sigma-1}{\sigma}}\left(\frac{1-\beta}{1-\alpha-\beta}\right)+\Gamma\left(\frac{1-\beta}{1-\alpha-\beta}\right)$$

$0<y<\infty$ means we can divide by $y^{-\frac{\sigma-1}{\sigma}}$ without changing the inequality. So we must show

$$1+\alpha(\sigma-1) > \frac{1-\beta}{1-\alpha-\beta} \Leftrightarrow \alpha(\sigma-1) > \frac{\alpha}{1-\alpha-\beta} \Leftrightarrow \sigma > 1+\frac{1}{1-\alpha-\beta} = \sigma^{\dagger}$$

The last statement is true, given the High Substitutability condition assumed in this case, so reversing the chain of implications means we have proved (B12) > (38).

(iii) Proof that the $\Delta\ln(n_t)=0$ locus is locally steeper than the development path:

The $\Delta\ln(n_t)=0$ locus in (N,y)-space is the curve:

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-1\right)}{\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-\frac{1-\beta}{1-\alpha-\beta}\right)} = \frac{\sigma-1}{\sigma} \tag{B13}$$

To compute the log-slope, $\frac{\Delta\ln(N)}{\Delta\ln(y)}$, of this locus, first set the difference of cross-products = 0:

$$\sigma\left(1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-1\right)-$$
$$(\sigma-1)\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-\frac{1-\beta}{1-\alpha-\beta}\right)=0 \tag{B14}$$

To expand Δ(B14) = 0, we will use these two first differences:

$$\Delta\left[\Gamma y^{\frac{\sigma-1}{\sigma}}\right] = \Gamma\frac{\sigma-1}{\sigma}y^{\frac{-1}{\sigma}}\Delta y = \Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y), \qquad \text{and}$$

$$\Delta\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}\right) = \Gamma^{\frac{1}{m}}\left(\frac{\sigma-1}{m\sigma}y^{\frac{\sigma-1}{m\sigma}-1}N^{\frac{-1}{m}}\Delta y - \frac{1}{m}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}-1}\Delta N\right)$$

$$= \frac{\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right) = \frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right).$$

$$\Rightarrow 0 = \Delta(\mathrm{M7}) = \sigma\left(1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\sigma\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}(n-1)\Delta\ln(y)$$

$$-(\sigma-1)\Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)\sigma(1-\beta)\left(n - \frac{1-\beta}{1-\alpha-\beta}\right)$$

$$-(\sigma-1)\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \sigma\left(1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)\Gamma y^{\frac{\sigma-1}{\sigma}}(n-1)\Delta\ln(y)$$

$$-(\sigma-1)^2(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\left(n - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(y)$$

$$-(\sigma-1)\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \left[1 + \alpha(\sigma-1) + \frac{(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{1-\alpha-\beta} - (\sigma-1)\left(1 + \Gamma y^{\frac{\sigma-1}{\sigma}}\right)(1-\beta)\right]\frac{\sigma n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left(\frac{1}{1-\alpha-\beta}\right)(n-1)(\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$-(\sigma-1)\left(n - \frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= \Bigg[1 + \alpha(\sigma - 1) - (\sigma - 1)(1 - \beta) + \frac{(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{1 - \alpha - \beta} - (\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Bigg]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left[\left(\frac{1}{1 - \alpha - \beta}\right)(n - 1) - (\sigma - 1)\left(n - 1 - \frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= \left[1 - (\sigma - 1)(1 - \alpha - \beta) - \left(\sigma - 1 - \frac{1}{1 - \alpha - \beta}\right)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \left[-(1 - \alpha - \beta)(\sigma - \sigma^{\dagger}) - (\sigma - \sigma^{\dagger})(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left[-(\sigma - \sigma^{\dagger})(n - 1) + (\sigma - 1)\left(\frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= -\left[(1 - \alpha - \beta) + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{n}{m}(\sigma - 1)\Delta\ln(y)$$

$$-\left[(\sigma - \sigma^{\dagger})(n - 1) - (\sigma - 1)\left(\frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$+\left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{\sigma n}{m}\Delta\ln(N)$$

$$\Rightarrow \frac{\Delta\ln(N)}{\Delta\ln(y)} = \frac{\left[(1-\alpha-\beta)+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$

is the log-slope of the $\Delta\ln(n_t) = 0$ locus in (*y*,*N*) space.

We next show that the log-slope of the locus exceeds the path's slope, i.e from (B10):

$$\frac{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}} - \frac{\sigma-1}{\sigma} > 0$$

So we need to show the following:

$$\left\{\left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{n}{m}(\sigma - 1) + \left[(\sigma - \sigma^{\dagger})(n - 1) - (\sigma - 1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right\}\sigma > \left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})(\sigma - 1)\frac{\sigma n}{m}$$

$$\Leftrightarrow \left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}} > \left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})(\sigma-1)\frac{n}{m}$$

$$\Leftrightarrow \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}} > 0$$

$$\Leftrightarrow (\sigma-\sigma^{\dagger})(n-1) = \left(\sigma-1-\frac{1}{1-\alpha-\beta}\right)(n-1) > (\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow (\sigma-1)\left(n-1-\frac{\alpha}{1-\alpha-\beta}\right) > \frac{n-1}{1-\alpha-\beta}$$

$$\Leftrightarrow \sigma-1 > \frac{\frac{n-1}{1-\alpha-\beta}}{n-1-\frac{\alpha}{1-\alpha-\beta}} = \frac{n-1}{(n-1)(1-\alpha-\beta)-\alpha} \quad \Leftrightarrow \sigma > 1+\frac{n-1}{(n-1)(1-\alpha-\beta)-\alpha} = 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n-1)}}$$

We now use the result that:

$$1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n_\infty-1)}} = 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\sigma-1-\frac{1}{1-\beta}\right)}{\sigma-1-\frac{1}{1-\alpha-\beta}}-1\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}-1\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)(1-\beta)-(\sigma-1)(1-\alpha-\beta)}{(\sigma-1)(1-\alpha-\beta)-1}\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)\alpha}{(\sigma-1)(1-\alpha-\beta)-1}\right]}} = 1+\frac{1}{1-\alpha-\beta-\frac{(\sigma-1)(1-\alpha-\beta)-1}{(\sigma-1)}}$$

$$= 1+\frac{\sigma-1}{(\sigma-1)(1-\alpha-\beta)-(\sigma-1)(1-\alpha-\beta)+1} = \sigma$$

Hence we need to show $1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n-1)}} < 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n_\infty-1)}} = \sigma$, and this last statement is true because $n_\infty := \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\sigma-1-\frac{1}{1-\beta}\right)}{\sigma-1-\frac{1}{1-\alpha-\beta}}$ was shown earlier to be the *minimum* value of *n* on the $\Delta\ln(n_t) = 0$ locus, namely its asymptotic value, which is exceeded at any finite point on the locus.

Derivation in Appendix B6

Algebra for Proposition 3

As shown in the Proof of Lemma 1, showing

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)}>\frac{\sigma-1}{\sigma}$$

is the same as showing:

$$\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$>(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

We prove this inequality is true by showing the [LHS] > 0 and the RHS < 0 as follows:

$$[\text{LHS}]=\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$=\left(\frac{1-\beta}{1-\alpha-\beta}-(\sigma-1)(1-\beta)\right)\bar{n}-\left(\frac{1-\beta}{1-\alpha-\beta}\right)+(\sigma-1)(1-\beta)\left(\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$=\left[\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\bar{n}-\left(\frac{1}{1-\alpha-\beta}\right)+(\sigma-1)\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right](1-\beta)$$

which (because $\sigma-1<\frac{1}{1-\alpha-\beta}$ and $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$>\left[\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\frac{1-\beta}{1-\alpha-\beta}+\left(\sigma-1-\frac{1}{1-\beta}\right)\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right](1-\beta)$$

$$=\left(\frac{1}{1-\alpha-\beta}-\frac{1}{1-\beta}\right)\frac{(1-\beta)^2}{1-\alpha-\beta}>0.$$

$$\text{RHS}=\ (1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

$$=\ [(1-\alpha-\beta)(\sigma-1)-1]\bar{n}-(1-\beta)\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)+1$$

which (again because $\sigma-1<\frac{1}{1-\alpha-\beta}$ and $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$<\ [(1-\alpha-\beta)(\sigma-1)-1]\frac{1-\beta}{1-\alpha-\beta}-(1-\beta)\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)+1$$

$$= [(1-\alpha-\beta)-(1-\beta)](\sigma-1)\frac{1-\beta}{1-\alpha-\beta}-\frac{1-\beta}{1-\alpha-\beta}+1$$

$$= -\alpha(\sigma-1)\frac{1-\beta}{1-\alpha-\beta}-\frac{\alpha}{1-\alpha-\beta}<0.$$

Derivations in Appendix B7

Derivation of (B17)

$$E_{S,t}(p_t,N_{S,t})=\left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}}L_{S,t}(p_t)\,p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t) \tag{16}$$

$$\Rightarrow E_{S,t}^{\frac{\alpha}{1-\beta}}=\left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}L_{S,t}^{\frac{\alpha}{1-\beta}}\,p_{S,t}{}^{\frac{\alpha}{(1-\beta)(1-\alpha-\beta)}}$$

Inserting this into (B16) gives

$$n_{S,t}=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}^{\frac{1}{h(1-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{\alpha}{h(1-\beta)}}p_{S,t}^{\frac{\alpha}{h(1-\beta)(1-\alpha-\beta)}}L_{S,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}^{\frac{1-\alpha-\beta+\alpha}{h(1-\beta)(1-\alpha-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{1-\beta}{h(1-\beta)}}$$

$$\Rightarrow n_{S,t}=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}{}^{\frac{1}{h(1-\alpha-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{1}{h}} \tag{B17}$$

Derivations of (41) and (42):

$$\text{(A2)\& (A5)}\Rightarrow p_{M,t}^{\frac{1}{h(1-\beta)}}L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma y_t^{-\frac{1}{\sigma}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\beta)}}\left(\frac{L_t\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left[y_t^{-\frac{1}{\sigma}}y_t^{\frac{\sigma-1}{\sigma}\frac{1}{\sigma-1}}\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\beta)}}\left(\frac{L_t}{y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma}\right)^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\left(\frac{1}{\sigma-1}\frac{1}{[h(1-\beta)]}-\frac{1-\alpha-\beta}{h(1-\beta)}\right)}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\frac{1-\alpha-\beta}{h(\sigma-1)(1-\beta)}}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{-\left[\frac{(\sigma-\sigma^{\dagger})(1-\alpha-\beta)}{h(\sigma-1)(1-\beta)}\right]}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$(\text{A1})\&\ (\text{A4}) \Rightarrow p_{S,t}^{\frac{1}{h(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{h}}$$

$$= \left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\alpha-\beta)}} \left[L_t\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-1}\right]^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\left[\frac{1}{(\sigma-1)(1-\alpha-\beta)}-1\right]\frac{1}{h}} L_t^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\left(\frac{1}{1-\alpha-\beta}-\sigma+1\right)\frac{1}{h(\sigma-1)}} L_t^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{(\sigma-\sigma^{\dagger})}{h(\sigma-1)}} L_t^{\frac{1}{h}}$$

Then substitute the above into (B16) and (B17) respectively:

$$n_{M,t} = \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} p_{M,t}^{\frac{1}{h(1-\beta)}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$
$$= \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{-\left[\frac{(\sigma-\sigma^{\dagger})(1-\alpha-\beta)(\sigma-\sigma^{\dagger})}{h(\sigma-1)(1-\beta)}\right]} L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} \quad (41)$$

$$n_{S,t} = \lambda N_{S,t-1}^{\frac{\nu+\mu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma-\sigma^{\dagger}}{h(\sigma-1)}} L_t^{\frac{1}{h}} \quad (42)$$

Derivation of (B20)

From (14),

$$Y_{M,t} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}} \Longrightarrow \frac{Y_{M,t}}{L_{M,t}} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{-\alpha}{1-\beta}}$$

which, substituting (A2) for $p_{M,t}$ and (A5) for $L_{M,t}$,

$$= \frac{1}{\beta} N_{M,t}\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}}\left(\frac{L_t\Gamma}{y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma}\right)^{\frac{-\alpha}{1-\beta}}$$

$$= \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}\left(\frac{\beta}{1-\beta}\right)+\frac{\alpha}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_t^{\frac{-\alpha}{1-\beta}}$$

$$= \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_t^{\frac{-\alpha}{1-\beta}} \quad (\text{B20})$$

Derivation of (B22)

Substituting (16) for $E_{S,t}$ into (15) and rearranging:

$$E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}\, p_{S,t}{}^{\frac{1}{1-\alpha-\beta}} \tag{16}$$

$$Y_{S,t} = \frac{1}{\beta} N_{S,t} p_{S,t}^{\frac{\beta}{1-\beta}} E_{S,t}^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}} \tag{15}$$

$$\Rightarrow Y_{S,t} = \lambda N_{S,t} p_{S,t}{}^{\frac{\beta}{1-\beta}} \left[\left(\frac{N_{S,t}}{\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}\, p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}\right]^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

(Powers on: $N_{S,t}$: $\frac{1-\alpha-\beta}{1-\alpha-\beta} + \frac{1-\beta}{1-\alpha-\beta}\frac{\alpha}{1-\beta} = \frac{1-\beta}{1-\alpha-\beta}$; $p_{S,t}$: $\frac{\beta(1-\alpha-\beta)+\alpha}{(1-\beta)(1-\alpha-\beta)} = \frac{\beta(1-\beta)+\alpha(1-\beta)}{(1-\beta)(1-\alpha-\beta)} = \frac{\alpha+\beta}{1-\alpha-\beta}$)

$$= \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{\alpha+\beta}{1-\alpha-\beta}} L_{S,t}$$

and then using (A1) for $p_{S,t}$

$$\Rightarrow \frac{Y_{S,t}}{L_{S,t}} = \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} \left[(1-\gamma)^{\frac{\sigma}{\sigma-1}} \left(1 + \Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

$$= \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} \left(1 + \Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \tag{B22}$$

Derivations in Appendix B8

Full algebra for comparative statics results stated after (B26).

Calculation of $\frac{\partial ln p_{S,t}(y_t)}{\partial y_t}$

$$\text{(A1)} \Rightarrow \ln\left(p_{S,t}(y_t)\right) = \frac{\sigma}{\sigma-1}\ln(1-\gamma) + \frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$\Rightarrow \frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} = \frac{\frac{\gamma}{1-\gamma}\frac{\sigma-1}{\sigma} y_t^{\frac{-1}{\sigma}}}{(\sigma-1)\left(\frac{\gamma}{1-\gamma} y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} = \frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln p_{M,t}(y_t)}{\partial y_t}$

$$\text{(A2)} \Rightarrow \ln\left(p_{M,t}(y_t)\right) = \frac{\sigma}{\sigma-1}\ln(1-\gamma) + \frac{1}{\sigma-1}\ln\left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)$$

$$\Rightarrow \frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} = \frac{-\frac{\sigma-1}{\sigma} y_t^{-\frac{\sigma-1}{\sigma}} y_t^{-1}}{(\sigma-1)\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right)} = \frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t}$

$$\text{(A4)} \Rightarrow \ \ln\left(L_{S,t}(\,y_t)\right) = \ln(L_t) - \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$\Rightarrow \frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} = -\frac{\Gamma \frac{\sigma-1}{\sigma} y_t^{\frac{-1}{\sigma}}}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}} = \frac{-(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t}$

$$\text{(A5)} \Rightarrow \ \ln\left(L_{M,t}(\,y_t)\right) = \ln(L_t) + \ln(\Gamma) - \ln\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right) \Rightarrow \frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t}$$

$$= \frac{\sigma-1}{\sigma} \frac{y_t^{-\frac{\sigma-1}{\sigma}} y_t^{-1}}{\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right)} = \frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Simplification of $\frac{\partial f}{\partial y_t}$

$$\frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right)\frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$+ \left(\frac{1-\alpha-\beta}{1-\beta}\right)\frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} + \frac{(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)} - \frac{1}{y_t}$$

$$= \frac{-(1-\alpha-\beta)\beta + (1-\alpha-\beta)^2(\sigma-1) - (1-\beta)(1-\alpha-\beta)\sigma}{(1-\beta)(1-\alpha-\beta)\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$+ \frac{[-(1-\beta)(\alpha+\beta) + (1-\beta)(1-\alpha-\beta)(\sigma-1) - (1-\beta)(1-\alpha-\beta)\sigma]\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{(1-\beta)(1-\alpha-\beta)\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$= \frac{-(1-\alpha-\beta)(\beta+1-\alpha-\beta)+(1-\alpha-\beta)\sigma[(1-\alpha-\beta)-(1-\beta)]-(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{(1-\beta)(1-\alpha-\beta)\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)}$$

$$= -\frac{(1-\alpha-\beta)(1-\alpha+\sigma\alpha)+(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\sigma(1-\beta)(1-\alpha-\beta)y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} \tag{B27}$$

Derivation of $f\left(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right)$

$$\begin{aligned} f\left(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right) & \\ \equiv \ln\left(\bar{N}_{M,t}\right) - \left(\frac{1-\beta}{1-\alpha-\beta}\right)\ln\left(\bar{N}_{S,t} +\right)\frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) & \\ -\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right) + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) & \\ -\left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right) & \\ -\ln\left(L_{S,t}(y_t)\right) - \ln(y_t) = 0 & \end{aligned} \tag{B25}$$

Insert

$$\ln(y_t) = \sigma\ln(\Gamma) + (1-\beta)\sigma\left(\ln\left(N_{M,t}\right) - \ln\left(N_{S,t}\right)\right) - \alpha\sigma\ln(e_t)$$

$$\begin{aligned} \Rightarrow f\left(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right) & \\ \equiv \ln\left(\bar{N}_{M,t}\right) - \left(\frac{1-\beta}{1-\alpha-\beta}\right)\ln\left(\bar{N}_{S,t}\right) + \frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) & \\ -\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right) + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) & \\ +\left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right) - \ln\left(L_{S,t}(y_t)\right) - \sigma\ln(\Gamma) & \\ -(1-\beta)\sigma\left(\ln\left(N_{M,t}\right) - \ln\left(N_{S,t}\right)\right) + \alpha\sigma\ln(e_t) = 0 & \end{aligned}$$

$$\begin{aligned} = -\sigma\ln(\Gamma) + [1-(1-\beta)\sigma]\ln\left(\bar{N}_{M,t}\right) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln\left(\bar{N}_{S,t}\right) & \\ +\frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right) + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) & \\ -\left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right) - \ln\left(L_{S,t}(y_t)\right) & \\ +\alpha\sigma\ln(e_t) = 0 & \end{aligned}$$

Now using (29):

$$y_t = \Gamma^\sigma \bar{N}_t^{(1-\beta)\sigma} e_t^{-\alpha\sigma} \Rightarrow \Gamma y_t^{\frac{\sigma-1}{\sigma}} = \Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)}$$

$$\Rightarrow \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)=\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right)$$

gives these versions of the logarithmic forms seen at intermediate stages of calculating $\partial\ln\left(p_{S,t}(y_t)\right)/\partial y_t$, $\partial\ln\left(p_{M,t}(y_t)\right)/\partial y_t$, $\partial\ln\left(L_{S,t}(y_t)\right)/\partial y_t$ and $\partial\ln\left(L_{M,t}(y_t)\right)/\partial y_t$ respectively:

$$\ln\left(p_{M,t}(e_t)\right)=-\frac{1}{\sigma}\ln(y_t)+\frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)+\frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$=-[\ln(\Gamma)+(1-\beta)\ln(\bar{N}_t)-\alpha\ln(e_t)]+\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right)+\frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$\ln\left(p_{S,t}(e_t)\right)=\frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)+\frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$=\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right)+\frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$\ln\left(L_{M,t}(e_t)\right)=\ln(L_t)+\ln(\Gamma)+\frac{\sigma-1}{\sigma}\ln(y_t)-\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)$$

$$=\ln(L_t)+\ln(\Gamma)+(\sigma-1)[\ln(\Gamma)+(1-\beta)\ln(\bar{N}_t)-\alpha\ln(e_t)]-\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right)$$

$$\ln\left(L_{S,t}(e_t)\right)=\ln(L_t)-\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)$$

$$=\ln(L_t)-\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right)$$

$$\Rightarrow f\left(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right)$$

$$= -\sigma\ln(\Gamma) + [1-(1-\beta)\sigma]\ln\left(\bar{N}_{M,t}\right) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln\left(\bar{N}_{S,t}\right)$$
$$- \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]$$
$$+ \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\}$$
$$- \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\}$$
$$+ \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{\mathrm{E}}_{\mathrm{M}}) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right)$$
$$+ \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma)$$
$$+ (\sigma-1)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\}$$
$$- \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) - \ln(L_t)$$
$$+ \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \alpha\sigma\ln(e_t)$$

$$= -\sigma\ln(\Gamma) + [1-(1-\beta)\sigma]\ln\left(N_{M,t-1}\right) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln\left(N_{S,t-1}\right)$$
$$- \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]$$
$$+ \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\}$$
$$- \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\}$$
$$+ \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right)$$
$$+ \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma)$$
$$+ (\sigma-1)[ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\}$$
$$- \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) - \ln(L_t)$$
$$+ \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \alpha\sigma\ln(e_t)$$

and then using $\frac{\beta}{1-\beta} - \frac{\alpha+\beta}{1-\alpha-\beta} = -\frac{\alpha}{(1-\beta)(1-\alpha-\beta)}$ and $-\left(\frac{1-\alpha-\beta}{1-\beta}\right) + 1 = \frac{\alpha}{1-\beta}$ gives (B28).